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UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

MAIN EXAMINATION 2005

TITLE OF PAPER	:	MATHEMATICAL METHODS I (PAPER TWO)
COURSE NUMBER	:	E370(II)
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
		EACH QUESTION CARRIES 25 MARKS.
		MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS EIGHT PAGES, INCLUDING THIS PAGE.

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E370(II) MATHEMATICAL METHODS I (PAPER TWO)

Question one

Given the following system of linear equations :

$$\begin{cases} 2x + 3y + z = 4 \\ 5x - 3y + z = 7 \\ 8x + 9y - 3z = 2 \end{cases} \quad \text{i.e.,} \quad A\,X = b$$

(a) use *linsolve* command to find its solutions. (3 marks)

(b) use the method of Gauss elimination , i.e., use *addrow* and *backsub* commands on an augment matrix of A *and* b , to find its solutions. Compare them with that obtained in (a). (6 marks)

(c) use Cramer's rule to find its solutions. Compare them with that obtained in (a).

(5 marks)

(d) (i) use the method of Gauss - Jordan elimination , i.e., use *addrow* and *mulrow* commands on an augment matrix of A *and* I , to find the inverse matrix of A , i.e., A^{-1} . (8 marks)

(ii) find its solution by evaluating the matrix product of $A^{-1}\,b$. Compare them with that obtained in (a). (3 marks)

Question two

(a) Given the following scalar function $f = x\,y - y\,z$

(i) find the $grad \;\; f$ at the point $P:(2\,,0\,,7)$, (3 marks)

(ii) find the directional derivative of f at $P:(2\,,0\,,7)$ in the direction of

$\vec{a} = \vec{i} - 2\,\vec{j} + \vec{k}$. (3 marks)

(b) Given a vector field as $\vec{F} = \left[\, 3x^2 - 4\,x\,y\,, - 2\,x^2\,, 0 \,\right]$,

find the value of the following line integral $\int_C \vec{F} \bullet d\vec{r}$

(i) where C : straight line from point $P_1:(-2\,,-2)$ to $P_2:(+2\,,+2)$

on x-y plane , (5 marks)

(ii) where C : circular path from P_1 *to* P_2 in counterclockwise sense with

radius of $2\sqrt{2}$ and centred at the origin on x-y plane, i.e.,

$x = 2\sqrt{2}\,\cos(\theta)$, $y = 2\sqrt{2}\,\sin(\theta)$ and $\theta = \dfrac{5\pi}{4} \ldots \dfrac{9\pi}{4}$

(6 marks)

(iii) find $\vec{\nabla} \times \vec{F}$ then remark briefly about the results of (b)(i) and (b)(ii) .

(2 marks)

(c) Given the following integral $\int_{(1,0,2)}^{(3,-4,6)} (3\,y^2\,z^3\,d\,x + 6\,x\,y\,z^3\,d\,y + 9\,x\,y^2\,z^2\,d\,z)$,

show that the given integral is exact and then find its value . (6 marks)

Question three

(a) Given a vector field in spherical coordinate system as :

$$\vec{F} = \vec{e}_r \, r^2 \sin(\theta) + \vec{e}_\theta \, r^2 \cos(\theta) + \vec{e}_\phi \, r^2 \sin(\phi) \quad ,$$

(i) evaluate the closed surface integral $\oiint_S \vec{F} \bullet \vec{n} \, d\,A$ where

S : the closed surface of a sphere with radius $r = 3$ and centred at the origin ,

(i.e., $\vec{n} \, d\,A = \vec{e}_r \, 9 \sin(\theta) \, d\theta \, d\phi \quad , \quad 0 \le \theta \le \pi \quad , \quad 0 \le \phi \le 2\pi$)

(6 marks)

(ii) find $\vec{\nabla} \bullet \vec{F}$ and evaluate the volume integral of $\iiint_V (\vec{\nabla} \bullet \vec{F}) \, d\,v$ where V is the volume enclosed by given closed surface S in (a)(i) and

$d\,v = r^2 \sin(\theta) \, d\,r \, d\,\theta \, d\,\phi$. Compare results in (a)(i) and (a)(ii) and remark briefly on the divergence theorem .

(6 marks)

(b) Given a vector field in cylindrical coordinate system as :

$$\vec{G} = \vec{e}_\rho \, \rho^3 + \vec{e}_\phi \, \rho^2 (z + 1 - \cos(\phi)) + \vec{e}_z \, \rho \sin(\phi)$$

(i) evaluate the closed loop line integral $\oint_l \vec{G} \bullet d\,\vec{l}$ where

l : the circular closed loop in counterclockwise sense with radius 4 centred at the origin on x - y plane , i.e., $\rho = 4 \quad , \quad z = 0 \quad , \quad \phi = 0 \; to \; 2\pi$ and

$d\,\vec{l} = \vec{e}_\phi \, 4 \, d\,\phi$

(6 marks)

(ii) find $\vec{\nabla} \times \vec{G}$ and evaluate the surface integral $\iint_S (\vec{\nabla} \times \vec{G}) \bullet d\,\vec{s}$ where

S : the surface region enclosed by the given closed loop l in (b)(i) , i.e.,

$d\,\vec{s} = \vec{e}_z \, \rho \, d\,\rho \, d\,\phi$

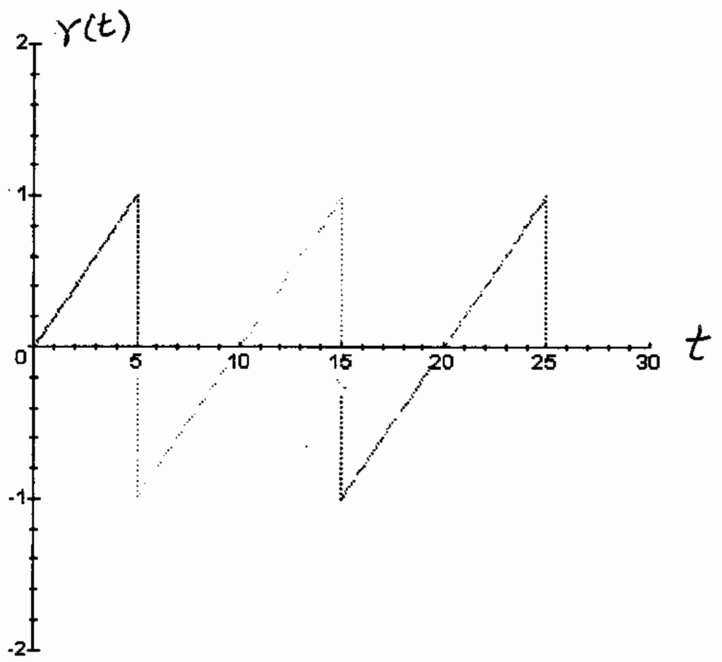
Compare results in (b)(i) and (b)(ii) and remark briefly on the Stokes's theorem .

(7 marks)

Question four

(a) Given the differential equation for a forced oscillations under a periodic jigsaw driving force $r(t)$ of period 10 as :

$$\frac{d^2\, y(t)}{d\,t^2} + 2\,\frac{d\,y(t)}{d\,t} + 20\,y(t) = r(t) \qquad \text{where} \quad r(t) \quad \text{is given as :}$$



(i) find the Fourier series of $r(t)$ up to the first 5 terms in its sine series and the first 5 terms in its cosine series . Plot it for $t = 0 \ to \ 30$ sec .

(6 marks)

(ii) find the steady - state solution corresponding to the first Fourier component of $r(t)$ obtained in (a)(i) . (8 marks)

(b) Any non-periodical function $f(x)$ $(-\infty < x < \infty)$ can be represented by a Fourier integral $f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]dw$ where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)dx \quad , \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)dx \ .$$

(i) Show that the following given integral on the left hand side represent the given function on the right hand side :

$$\int_0^\infty \frac{\omega^3 \sin(x\omega)}{\omega^2 + 4} d\omega = \begin{cases} -\dfrac{\pi}{2} e^{x} \cos(x) & if \quad x < 0 \\ \dfrac{\pi}{2} e^{-x} \cos(x) & if \quad x > 0 \end{cases}$$

(9 marks)

(ii) evaluate the values of the given integral in (b)(i) for $x = 2.5$. (2 marks)

Question five

The vibrations of a certain elastic string of length $L = 10$ and fixed at both ends, i.e., $x = 0$ and $x = 10$, are governed by the following one-dimensional wave equation :

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 9\frac{\partial^2 u(x,t)}{\partial x^2}$$

(a) Setting $u(x,t) = X(x)\,T(t)$ and applying the technique of separation of variables ,

(i) deduce the following two ordinary differential equations :

$$\begin{cases} \dfrac{d^2 X(x)}{d x^2} = -k^2 X(x) \cdots\cdots (1) \\ \dfrac{d^2 T(t)}{d t^2} = -9k^2 T(t) \cdots\cdots (2) \end{cases}$$

where $k\,(>0)$ is a separation constant , (4 marks)

(ii) for any value of k , show that the general solution for equations (1) & (2) can be written as : $T_k(t) = (C_k \cos(3kt) + D_k \sin(3kt))$,

$X_k(x) = (A_k \cos(kx) + B_k \sin(kx))$ and $u_k(x,t) = X_k(x)\,T_k(t)$

where A_k , B_k , C_k and D_k are arbitrary constants. (3 marks)

(iii) applying the given fixed end boundary conditions , i.e., $u_k(0,t) = 0 = u_k(10,t)$, deduce that $A_k = 0$ and

$k = \dfrac{n\pi}{10}$, $n = 1, 2, \cdots$ (4 marks)

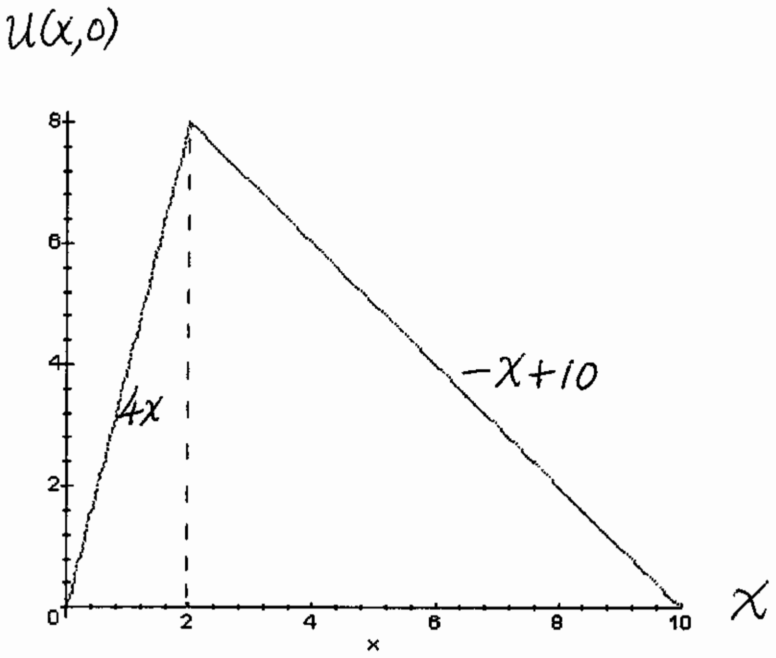
Question five (continued)

(b) after satisfying the fixed end boundary conditions and re-indexing k as n, the general solution can be written as $u(x,t)=\sum_{n=1}^{\infty} u_n(x,t)$ where

$$u_n(x,t)=(C_n\cos(\frac{3n\pi}{10}t)+D_n\sin(\frac{3n\pi}{10}t))\sin(\frac{n\pi}{10}x) ,$$

(i) if given zero initial vibration speed, i.e, $\left.\frac{\partial u_n(x,t)}{\partial t}\right|_{t=0}=0$, deduce that $D_n=0$ (3 marks)

(ii) if given the initial position $u(x,0)$ as



find the values of C_n for $n=1$ *to* 10. Then for $t=1$ and $t=2$, plot $\sum_{n=1}^{10} u_n(x,t)$ for $x=0$ *to* 10. Show them in a single display. (11 marks)