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UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

MAIN EXAMINATION 2005

TITLE OF PAPER	:	MATHEMATICAL METHODS II (PAPER ONE)
COURSE NUMBER	:	E470(I)
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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E470(I) MATHEMATICAL METHODS II (PAPER ONE)

Question one

(a) **Given the following complex function** $w = z\,e^{-z^2} - 6\,z^3 + 5$ **where** $z = x + iy$ **and** $w \equiv u(x,y) + i\,v(x,y)$,

(i) **find its** $u(x,y)$ *and* $v(x,y)$, **(4 marks)**

(ii) **check for its analyticity ,** **(4 marks)**

(iii) **plot the mapped image of** $u(x,y) = 0$, $u(x,y) = 2$, $u(x,y) = 4$ **and** $u(x,y) = 6$ **curves onto the** $z - plane$ **and show them in one display in** $z - plane$ **for** $x = -2$ *to* $+2$ **and** $y = -2$ *to* $+2$.

(6 marks)

(iv) **use** *conformal* **command to plot the mapped region in** $w - plane$ **corresponding to a narrow strip rectangular region in** $z - plane$ **with vertices of** $z = 0$, $z = 5$, $z = 0.001\,i$ **and** $z = 5 + 0.001\,i$.

(3 marks)

(b) **(i)** **Determine the value of** a **such that** $u(x,y) = x^4 + a\,x^2\,y^2 + y^4 - 5\,x$ **is a harmonic ,** **(4 marks)**

(ii) **and then find its conjugate harmonic** $v(x,y)$. **(4 marks)**

Question two

(a) Evaluate the value of the following complex line integral $\int_C (z^2 e^z - 4z)\, dz$ if

(i) C : the shortest path from $z = 3$ to $z = 3i$, (5 marks)

(ii) C : the counter clockwise circular path from $z = 3$ to $z = 3i$ with the centre of the circle at the origin .

Compare the answer here with that obtained in (a)(i) and make brief comment . (7 marks)

(b) Given the following complex function $f(z)$ as :

$$f(z) = \frac{2}{z + 3 - 4i} - \frac{5}{z + 3 + 4i}$$

(i) find its convergent series expansion about $z = -3 - i$ for all the values of z in the domain of $|z + 3 + i| < 3$. (3 marks)

(ii) find its convergent series expansion about $z = -3 - i$ for all the values of z in the domain of $3 < |z + 3 + i| < 5$, (6 marks)

(iii) find its convergent series expansion about $z = -3 - i$ for all the values of z in the domain of $|z + 3 + i| > 5$. (4 marks)

Question three

(a) **Find the centre and the radius of convergence of the following power series :**

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+2}\,(n!)^2}{(2n)!}\,(z+5-3i)^n$$ **(5 marks)**

(b) **Evaluate the value of the following contour integral**

$$\oint_C \frac{e^{-2z}}{(z^2+9)(z^2-4z+5)}\,dz$$ **where**

C : **the boundary of the circle** $|z+2-3i| = 3$ **, and looping in counterclockwise sense .** **(7 marks)**

(c) **Given the following definite integral :**

$$\int_0^{2\pi} \frac{\sin(\theta) - \cos(2\theta)}{25 - 24\cos(\theta)}\,d\theta$$

(i) **use** int **command to find its value ,** **(3 marks)**

(ii) **convert it to a complex contour integral , evaluate the value of this contour integral . Compare it with that obtained in (i) .** **(10 marks)**

Question four

(a) **Given the following two improper integrals :**

$$\int_{-\infty}^{\infty} \frac{\sin(k\,x)}{x^2 + x + 1}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\cos(k\,x)}{x^2 + x + 1}\,dx \quad \text{where } k \text{ is a positive constant}$$

(i) **convert them into the real and imaginary part of a complex contour integral respectively . Justify your choice of the contour .** **(3 marks)**

(ii) **Integrate the converted contour integral in (a)(i) by the method of residue integration. Evaluate the values of the given integrals if** $k = 5$ **.**

(10 marks)

(b) **Find the Cauchy principal value of the following integral :**

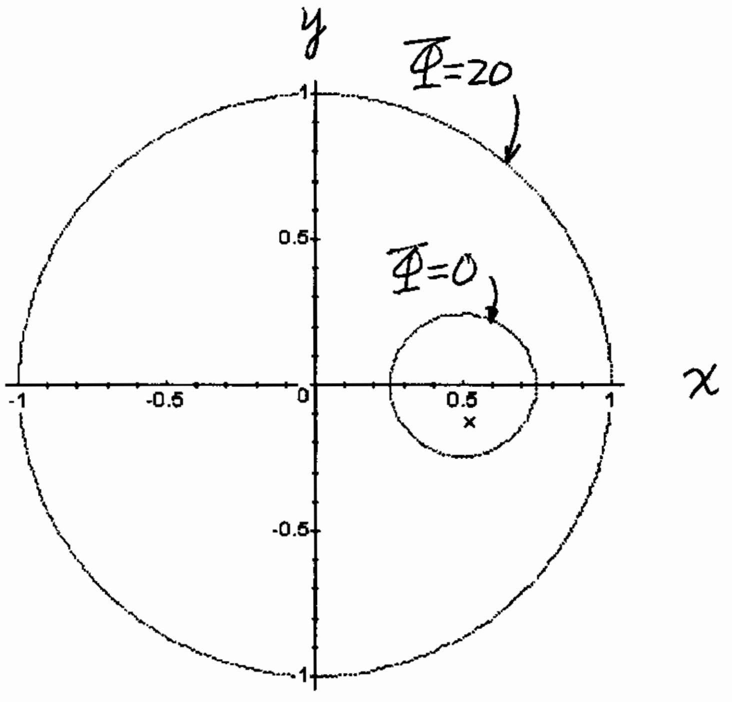
$$\int_{-\infty}^{\infty} \frac{x + 3}{x^3 - x^2 - x - 15}\,dx$$ **(7 marks)**

(c) **Evaluate by the method of residue integration the following improper integral :**

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 - 2\,x + 5)^2}\,dx$$ **(5 marks)**

Question five

A pair of long, non-coaxial, circular cross-section conductors is statically charged such that the inner conductor (radius of $\frac{1}{4}$ and centred at $\left(x = \frac{1}{2}, y = 0 \right)$) is at zero potential, i.e., $\Phi = 0$ volt, while the outer conductor (radius of 1 and centred at origin) is maintained at $\Phi = 20$ volts as shown in the diagram below :



Use the linear fractional transformation of the form $w = \dfrac{z-b}{bz-1}$ to transform the above given non-coaxial circles in $z-plane$ $\left(z = x + i\,y\right)$ to two coaxial circles in $w-plane$ $\left(w = u + i\,v\right)$,

(a) show that $w = \dfrac{z-b}{bz-1}$ maps the unit circle in $z-plane$, i.e., $z = e^{i\theta}$, onto the unit circle in $w-plane$ for any real value of b, (5 marks)

Question five (continued)

(b) **find the appropriate value of** b **such that the inner circle of radius** $\frac{1}{4}$ **maps to a coaxial circle of radius** $r_0\,(< 1)$ **. Find also the value of** r_0 **.** **(8 marks)**

(c) **since the general solution for coaxial conductors can be written as** $\Phi = k_1 \ln(|w|) + k_2$ **, determine the values of** k_1 *and* k_2 **from the given boundary conditions .** **(4 marks)**

(d) **plot the equal potential surfaces** $\Phi = 0$, $\Phi = 10$ **and** $\Phi = 20$ **in** $z-$ *plane* **and show them in a single display .** **(8 marks)**