UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

SUPPLEMENTARY EXAMINATION 2006

TITLE OF PAPER	:	**MATHEMATICAL METHODS I (PAPER TWO)**
COURSE NUMBER	:	**E370(II)**
TIME ALLOWED	:	**THREE HOURS**
INSTRUCTIONS	:	**ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.**

THIS PAPER HAS SIX PAGES, INCLUDING THIS PAGE.

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Question one

(a) Given the following scalar function $f = 4x^2 - 9y^2$

(i) plot $f = 10$, $f = 20$ and $f = 30$ on x-y plane for $x = -5$ *to* 5 and $y = -5$ *to* 5 show them in one display ,

(4 marks)

(ii) find the $grad\ f$ at the point $P:(-3,3)$, (3 marks)

(iii) find the directional derivative of f at $P:(-3,3)$ in the direction of $\vec{a} = 4\vec{i} + 6\vec{j}$.

(4 marks)

(b) Given a vector field $\vec{F} = \left[6y^2z^3 - 7e^x \ , 12xyz^3 \ , 18xy^2z^2 + \frac{8}{z}\right]$,

find $curl(\vec{F})$ and show that it is a conservative vector field and thus find its scalar potential .

(6 marks)

(c) For any vector fields $\vec{G} = \left[G_x(x,y,z) \ , \ G_y(x,y,z) \ , \ G_z(x,y,z)\right]$ and any scalar field $f(x,y,z)$ show that

$$div(f\,\vec{G}) \equiv \vec{G} \bullet grad(f) + f\left(div(\vec{G})\right) \ .$$

(8 marks)

Question two

(a) Given a vector field $\vec{F} = \left[e^{x}, e^{-y}, e^{z}\right]$,

find the value of the following line integral $\int_C \vec{F} \bullet d\vec{r}$ where

$C : \vec{r} = [\, t \,, t^2 \,, t \,]$

from $(0, 0, 0)$ to $(1, 1, 1)$. (8 marks)

(b) Given the following integral $\int_{(0,2,3)}^{(1,1,1)} (y z \sinh(xz)dx + \cosh(xz)dy + x y \sinh(xz)dz)$

(i) show that the form under the integral sign is exact in space, (4 marks)

(ii) evaluate the given integral . (4 marks)

(c) Using Green's theorem, evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ counterclockwise around the boundary C of the region R, where $\vec{F} = \vec{i}\, e^{-y} + \vec{j}\, e^{x}$ and

C : the ellipse $25x^2 + 9y^2 = 225$. (9 marks)

Question three

(a) Given a vector field $\vec{F} = \left[4xy \ , \ 2x^2 \ , \ 0\right]$ and a surface region

$S: \ \vec{r} = \left[\cosh(u) \ , \ \sinh(u) \ , v\right] \ , \ 0 \le u \le 2 \ , \ -3 \le v \le 3$,

(i) plot $\left[x = \cosh(u) \ , \ y = \sinh(u)\right]$ line segment for $u = 0 \ to \ 2$ on $x-y \ plane$. Describe the given surface S and find its total surface area ,

(6 marks)

(ii) find the unit normal vector $\vec{n}$ of the given surface , (4 marks)

(iii) evaluate the surface integral $\iint_S \vec{F} \bullet \vec{n} dA$ where $dA = (du)(dv)$.

(7 marks)

(b) Given a vector field $\vec{G} = \left[r^3 \ , \ r^3 \sin\theta \ , \ r^3 \cos\phi\right]$ and a closed spherical surface $S \ : \ r = 4$,

(i) find $\vec{\nabla} \cdot \vec{G}$, (2 marks)

(ii) evaluate the closed surface integral $\oiint_S \vec{G} \bullet \vec{n} dA$ by using the divergence theorem . (note : the small volume element in spherical coordinate system is $dv = r^2 \sin\theta \left(dr\right)\left(d\theta\right)\left(d\phi\right)$) (6 marks)

Question four

(a) Given a periodic function $f(x)$ of period 8 as

$$f(x) = \begin{cases} x & \text{if } -4 < x < 0 \\ x^2 & \text{if } 0 < x < 4 \end{cases}$$

(i) find the Fourier series of $f(x)$, (8 marks)

(ii) plot the first ten partial sums of the Fourier series in (i) (i.e., the first five partial sums of its cosine series plus the first five partial sums of its sine series) for $x = -4 \ \ to \ \ +4$. Also plot the given $f(x)$ for $x = -4 \ \ to \ \ +4$. Show them in a single display. (6 marks)

(b) Any non-periodical function $f(x)$ $(-\infty < x < \infty)$ can be represented by a Fourier integral $f(x) = \int_0^{\infty} [A(w)\cos(wx) + B(w)\sin(wx)]dw$ where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)dx \quad , \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)dx \ .$$

(i) Show that the following given integral on the left hand side represent the given function on the right hand side :

$$\int_0^{\infty} \frac{1-\cos(\pi\,\omega)}{\omega}\sin(x\,\omega)\,dw = \begin{cases} \dfrac{\pi}{2} & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$ (9 marks)

(ii) evaluate the values of the given integral in (i) for $x = 1.4$ and $x = 5.3$.

(2 marks)

Question five

The vibrations of a certain elastic string of length $L = 10$ and fixed at both ends, i.e., $x = 0$ and $x = 10$, are governed by the following one-dimensional wave equation :

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 4\frac{\partial^2 u(x,t)}{\partial x^2}$$

(a) the general solution can be written as $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ where

$$u_n(x,t) = (A_n \cos(\frac{n\pi}{5}t) + B_n \sin(\frac{n\pi}{5}t)) \sin(\frac{n\pi}{10}x) \quad ,$$

(i) by direct substitution, show that $u_n(x,t)$ above satisfies the given wave equation, (5 marks)

(ii) show that at $x = 0$ and $x = 10$, $u_n(x,t) = 0$. (2 marks)

(b) if at $t = 0$, $u(x,t) = \begin{cases} 4x & if \quad 0 \le x \le 2 \\ -x + 10 & if \quad 2 \le x \le 10 \end{cases}$ and $\frac{\partial u(x,t)}{\partial t} = 0$,

(i) find the values of A_n and B_n for $n = 1$ *to* 10 , (13 marks)

(ii) for $t = 1$ and $t = 2$, plot $\sum_{n=1}^{10} u_n(x,t)$ for $x = 0$ *to* 10 . Show them in a single display . (5 marks)