UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

MAIN EXAMINATION 2006

TITLE OF PAPER	**:**	**MATHEMATICAL METHODS I (PAPER TWO)**
COURSE NUMBER	**:**	**E370(II)**
TIME ALLOWED	**:**	**THREE HOURS**
INSTRUCTIONS	**:**	**ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.**

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E370(II) MATHEMATICAL METHODS I (PAPER TWO)

Question one

(a) From $\begin{cases} \vec{e}_\rho = \vec{e}_x \cos(\phi) + \vec{e}_y \sin(\phi) \\ \vec{e}_\phi = -\vec{e}_x \sin(\phi) + \vec{e}_y \cos(\phi) \end{cases}$ and $\begin{cases} \vec{e}_r = \vec{e}_\rho \sin(\theta) + \vec{e}_z \cos(\theta) \\ \vec{e}_\theta = \vec{e}_\rho \cos(\theta) - \vec{e}_z \sin(\theta) \end{cases}$

deduce that $\begin{cases} \vec{e}_x = \vec{e}_\rho \cos(\phi) - \vec{e}_\phi \sin(\phi) \\ \vec{e}_y = \vec{e}_\rho \sin(\phi) + \vec{e}_\phi \cos(\phi) \end{cases}$ and

$$\begin{cases} \vec{e}_x = \vec{e}_r \sin(\theta)\cos(\phi) + \vec{e}_\theta \cos(\theta)\cos(\phi) - \vec{e}_\phi \sin(\phi) \\ \vec{e}_y = \vec{e}_r \sin(\theta)\sin(\phi) + \vec{e}_\theta \cos(\theta)\sin(\phi) + \vec{e}_\phi \cos(\phi) \\ \vec{e}_z = \vec{e}_r \cos(\theta) \qquad - \vec{e}_\theta \sin(\theta) \end{cases}$$

(6 marks)

(b) Given P(8 , 300^0 , - 6) in cylindrical coordinate system, find its Cartesian and spherical coordinate . **(6 marks)**

(c) Given the following scalar function $f = x^3 + 2\,x\,y^2 - 5\,y\,z^2$,

(i) find the $\vec{\nabla} f$ at the point $P:(1,2,3)$, **(4 marks)**

(ii) find the directional derivative of f at $P:(1,2,3)$ in the direction of

$\vec{a} = \vec{e}_x\,(-2) + \vec{e}_y\,3 + \vec{e}_z\,(-1)$. **(4 marks)**

(d) Given any vector field $\vec{G}$ in spherical coordinate as

$\vec{G} = \left[G_r(r,\theta,\phi)\,,\,G_\theta(r,\theta,\phi)\,,\,G_\phi(r,\theta,\phi)\right]$, show that

$\vec{\nabla} \bullet (\vec{\nabla} \times \vec{G}) \equiv 0$. **(5 marks)**

Question two

(a) Given a vector field $\vec{F} = \left[7y^2 - 5z^2 + 3yz\,,14xy + 3xz\,,3xy - 10xz\right]$,

find $\vec{\nabla} \times \vec{F}$ and show that it is a conservative vector field and thus find its scalar potential . **(5 marks)**

(b) Given a vector field as $\vec{G} = \left[\; y^3 \;,\; 5xy^2 \;,\; 0\right]$,

find the value of the following line integral $\int_C \vec{G} \bullet d\vec{r}$

(i) where C : straight line from point $P_1 : (-1\,,-1)$ to $P_2 : (+3\,,+3)$ on x-y plane , **(5 marks)**

(ii) where C : from point $P_1 : (-1\,,-1)$ to $P_2 : (+3\,,+3)$ along a semicircular path with radius of $\sqrt{8}$ and centred at $(\,1\,,1\,)$ in counter clockwise sense , **(6 marks)**

(iii) find $\vec{\nabla} \times \vec{G}$ then remark briefly about the results of (b)(i) and (b)(ii) .

(2 marks)

(c) Given a surface region $S \;:\; x^2 + 3y^2 + 2z^2 = 6$,

(i) plot the given surface for $x = -3$ *to* $+3$, $y = -3$ *to* $+3$ and $z = -3$ *to* $+3$, **(3 marks)**

(ii) find the normal unit vector $\vec{n}$ on the given surface at the point $(1\,,1\,,1)$.

(4 marks)

Question three

(a) Given a vector field $\vec{F} = \left[2\,\rho^2\,,\rho\; z\cos(\phi)\,,3\,\rho\;(z+1)\right]$, and a closed surface S of a cylindrical tube of height 3 , cross-section radius 2 and its central axis coincides with z axis , i.e., $S = S_1 + S_2 + S_3$ where

$S_1\ :\ z = 0\ \ ,\ \ 0 \le \rho \le 2\ \ and\ \ 0 \le \phi \le 2\,\pi$,

$S_2\ :\ z = 3\ \ ,\ \ 0 \le \rho \le 2\ \ and\ \ 0 \le \phi \le 2\,\pi$ and

$S_3\ :\ \rho = 2\ \ ,\ \ 0 \le z \le 3\ \ and\ \ 0 \le \phi \le 2\,\pi$

(i) evaluate the value of the closed surface integral $\oiint_S \vec{F} \bullet \vec{n}\; d\,A$ **(8 marks)**

(ii) evaluate the value of the volume integral $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) d\,v$ where V is the volume bounded by the given S and $d\,v = \rho\,\left(d\,\rho\right)\left(d\,\phi\right)\left(d\,z\right)$.

(5 marks)

(b) Given a vector field $\vec{G} = \left[5\,y^3\,,-\,3\,x^2\;y\,,0\right]$ and a surface region S as

S: the circular disk $x^2 + y^2 \le 9$ and $z = 0$,

(i) evaluate the surface integral $\iint_S (\vec{\nabla} \times \vec{G}) \bullet \vec{n}\; d\,A$. **(6 marks)**

(ii) evaluate the closed line integral $\oint_C \vec{G} \bullet d\,\vec{l}$ where

C: the line boundary of the given surface region S in counter clockwise sense

Compare results in (i) and (ii) and remark briefly on the Stokes's Theorem .

(6 marks)

Question four

(a) Given a periodic function $f(x)$ of period 20, and one period ranging from -10 to 10 as

$$f(x) = \begin{cases} -x-10 & if \quad -10 \le x \le 0 \\ x & if \quad 0 \le x \le 5 \\ -x+10 & if \quad 5 \le x \le 10 \end{cases}$$

(i) find the Fourier series of $f(x)$, **(8 marks)**

(ii) plot the first ten partial sums of the Fourier series in (i) (i.e., the first five partial sums of its cosine series plus the first five partial sums of its sine series) for $x = -10 \;\; to \;\; 10$. Also plot the given $f(x)$ for $x = -10 \;\; to \;\; 10$. Show them in a single display. **(5 marks)**

(b) Any non-periodical function $f(x) \quad (-\infty < x < \infty)$ can be represented by a Fourier integral $f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]dw$ where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)dx \quad , \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)dx \ .$$

(i) Show that the following given integral on the left hand side represent the given function on the right hand side :

$$\int_0^\infty \frac{\cos(x\omega) + \omega \sin(x\omega)}{1+\omega^2} = \begin{cases} 0 \quad if \quad x < 0 \\ \dfrac{\pi}{2} \quad if \quad x = 0 \\ \pi e^{-x} \quad if \quad x > 0 \end{cases}$$ **(9 marks)**

(ii) evaluate the values of the given integral in (b)(i) for $x = -1.7$ and $x = 0.8$.

(3 marks)

Question five

The vibrations of a certain elastic string of length $L = 10$ metres and fixed at both ends, i.e., $x = 0$ and $x = 10$, are governed by the following one-dimensional wave equation :

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 16\frac{\partial^2 u(x,t)}{\partial x^2}$$

(a) the general solution can be written as $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ where

$$u_n(x,t) = (A_n \cos(\frac{2n\pi}{5}t) + B_n \sin(\frac{2n\pi}{5}t))\sin(\frac{n\pi}{10}x) \quad ,$$

(i) by direct substitution , show that $u_n(x,t)$ above satisfies the given wave equation , **(5 marks)**

(ii) show that at $x = 0$ and $x = 10$, $u_n(x,t) = 0$. **(2 marks)**

(b) if the string has zero initial speed , i.e., $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$, deduce that $B_n = 0$ for all the values of n , **(3 marks)**

(c) furthermore if the string has its initial position $u(x,0)$ given as

$$u(x,0) = \begin{cases} \frac{x}{2} & if \quad 0 \le x \le 4 \\ -\frac{x}{3} + \frac{10}{3} & if \quad 4 \le x \le 10 \end{cases}$$

(i) find the values of A_n for $n = 1$ *to* 10 , **(9 marks)**

(ii) for $t = 0$, $t = 1$ and $t = 2$, plot $\sum_{n=1}^{10} u_n(x,t)$ for $x = 0$ *to* 10.

Show them in a single display . **(6 marks)**