UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

SUPPLEMENTARY EXAMINATION 2007

TITLE OF PAPER	:	LINEAR ALGEBRA AND VECTOR CALCULUS
COURSE NUMBER	:	E372
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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E372 LINEAR ALGEBRA AND VECTOR CALCULUS

Question one

(a) Given the following scalar function $f = 4x^2 + y^2$

(i) plot $f = 10$, $f = 20$ and $f = 30$ on x-y plane for $x = -5$ *to* 5 and $y = -5$ *to* 5 show them in one display ,

(4 marks)

(ii) find the $grad\ f$ at the point $P{:}(1,-2)$, (3 marks)

(iii) find the directional derivative of f at $P{:}(1,-2)$ in the direction of $\vec{a} = 3\vec{i} - 5\vec{j}$. (4 marks)

(b) Given a vector field $\vec{F} = \left[e^z \ , \ 2y \ , \ x e^z\right]$,

find $curl(\vec{F})$ and show that it is a conservative vector field and thus find its scalar potential . (6 marks)

(c) For any vector fields $\vec{G} = \left[G_x(x,y,z) \ , \ G_y(x,y,z) \ , \ G_z(x,y,z)\right]$ and any scalar field $f(x,y,z)$ show that

$$div\left(f\,\vec{G}\right) \equiv \vec{G} \bullet grad\left(f\right) + f\left(div\left(\vec{G}\right)\right) .$$

(8 marks)

Question two

(a) Given a vector field $\vec{F} = \left[\cosh(x) \ , \ \sinh(y) \ , \ e^{z}\right]$,

find the value of the following line integral $\int_C \vec{F} \bullet d\vec{r}$ where

$C \ : \ \vec{r} = \left[t \ , \ t^2 \ , \ t^3 \right]$

from $(0,0,0)$ to $(2,4,8)$. (8 marks)

(b) Given the following integral $\int_{(0,-1,1)}^{(2,4,0)} e^{x-y+z^2} (dx - dy + 2z \ dz)$

(i) show that the form under the integral sign is exact in space , (4 marks)

(ii) evaluate the given integral . (4 marks)

(c) Using Green's theorem , evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ counterclockwise around the boundary C of the region R , where $\vec{F} = \vec{i} \ x y + \vec{j} \ x^2 y^2$ and

C : the circle $x^2 + y^2 = 25$. (9 marks)

Question three

(a) Given a vector field $\vec{F} = \left[x^2 \quad , \quad 0 \quad , \quad z^2 \right]$ and a closed surface S

where S : the surface of the box $|x| \le 1$, $|y| \le 2$, $|z| \le 3$

(i) find $\vec{\nabla} \bullet \vec{F}$, (2 marks)

(ii) evaluate the closed surface integral $\oiint_S \vec{F} \bullet d\vec{s}$ by using the divergence theorem . (7 marks)

(b) Given a vector field $\vec{G} = \left[2\rho\, z \quad , \rho^2 \quad , \rho^2 \cos\phi \right]$ in cylindrical coordinates

and a closed loop L where L : the circle of radius 2 on $z = 0$ plane , i.e.,

$\rho = 2$ and $0 \le \phi \le 2\pi$

(i) evaluate the value of closed line integral $\oint_L \vec{G} \bullet d\vec{l}$ where $d\vec{l} = \vec{e}_\phi\, 2\, d\phi$ here , (7 marks)

(ii) evaluate the value of surface integral $\iint_S \left(\vec{\nabla} \times \vec{G} \right) \bullet d\vec{s}$ where

S is the surface on $z = 0$ plane bounded by the given closed loop L , i.e.,

$d\vec{s} = \vec{e}_z\, \rho\, d\rho\, d\phi$, $0 \le \rho \le 2$ and $0 \le \phi \le 2\pi$. Compare this value with that obtained in (b)(i) and make a brief remark . (9 marks)

Question four

(a) Given a periodic function $f(x)$ of period 8 as

$$f(x) = \begin{cases} 4+x & \text{if } -4<x<0 \\ 4-x & \text{if } 0<x<4 \end{cases}$$

(i) find the Fourier series of $f(x)$, (8 marks)

(ii) plot the first ten partial sums of the Fourier series in (i) (i.e., the first five partial sums of its cosine series plus the first five partial sums of its sine series) for $x = -4$ *to* $+4$. Also plot the given $f(x)$ for $x = -4$ *to* $+4$. Show them in a single display. (6 marks)

(b) Any non-periodical function $f(x)$ $(-\infty < x < \infty)$ can be represented by a Fourier integral $f(x) = \int_0^{\infty}\left[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right]d\omega$ where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)dx \quad , \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)dx \ .$$

(i) Show that the following given integral on the left hand side represent the given function on the right hand side :

$$\int_0^{\infty} \frac{\cos(x\,\omega)}{1+\omega^2}\, d\omega = \begin{cases} \frac{\pi}{2} e^{-x} & \text{if} \quad x > 0 \\ \frac{\pi}{2} e^{+x} & \text{if} \quad x < 0 \end{cases}$$

(9 marks)

(ii) evaluate the values of the given integral in (i) for $x = -1$ and $x = +2$.

(2 marks)

Question five

The vibrations of a certain elastic string of length $L = 7$ and fixed at both ends, i.e., $x = 0$ and $x = 7$, are governed by the following one-dimensional wave equation :

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 9\frac{\partial^2 u(x,t)}{\partial x^2}$$

(a) the general solution can be written as $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ where

$$u_n(x,t) = (A_n \cos(\frac{3n\pi}{7}t) + B_n \sin(\frac{3n\pi}{7}t))\sin(\frac{n\pi}{7}x) \quad ,$$

(i) by direct substitution, show that $u_n(x,t)$ above satisfies the given wave equation, (5 marks)

(ii) show that at $x = 0$ and $x = 7$, $u_n(x,t) = 0$. (2 marks)

(b) if initially there is zero initial speed, i.e., $\left.\frac{\partial\, u_n(x,t)}{\partial t}\right|_{t=0} = 0$ for all values of n , deduce that $B_n = 0$, (3 marks)

(c) if at $t = 0$, $u(x,t) = \begin{cases} 5x & if \quad 0 \le x \le 2 \\ -2x + 14 & if \quad 2 \le x \le 7 \end{cases}$,

(i) find the values of A_n for $n = 1$ *to* 10 , (10 marks)

(ii) for $t = 1$ and $t = 2$, plot $\sum_{n=1}^{10} u_n(x,t)$ for $x = 0$ *to* 7 . Show them in a single display . (5 marks)