UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

MAIN EXAMINATION 2007/2008

TITLE OF PAPER	:	LINEAR ALGEBRA AND VECTOR CALCULUS
COURSE NUMBER	:	E372
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN. STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

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E372 Linear Algebra and Vector Calculus

Question one

Given the following matrix equation $AX = b$ where

$$A = \begin{pmatrix} 8 & 2 & -5 \\ -3 & 6 & 2 \\ 4 & -5 & -7 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \textit{and} \quad b = \begin{pmatrix} 4 \\ -35 \\ 23 \end{pmatrix}$$

(a) use the "linsolve" command to find the solution of X , **(3 marks)**

(b) use the Gauss elimination method to find the solution of X , **(6 marks)**

(c) use the Crammer's rule to find the solution of X , **(6 marks)**

(d) (i) use Gauss-Jordan elimination method to find A^{-1} , **(8 marks)**

(ii) use A^{-1} obtained in (d) (i) to find the solution of X . **(2 marks)**

Question two

Given the following differential equations for a coupled oscillator system as

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -5\, x_1(t) + 12\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 4\, x_1(t) - 18\, x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ *and* $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation $A\,X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -5 & 12 \\ 4 & -18 \end{pmatrix} \quad and \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} ,$$ **(4 marks)**

(b) (i) find the eigen frequencies of ω , **(4 marks)**

(ii) find the eigen vectors of X , **(4 marks)**

(c) (i) write down the general solutions of $x_1(t)$ *and* $x_2(t)$ in terms of the eigen frequencies and eigen vectors obtained in (b) , **(4 marks)**

(ii) if initial conditions are given as

$$x_1(0) = 3 \;,\quad x_2(0) = -2 \;,\quad \left.\frac{d\,x_1(t)}{dt}\right|_{t=0} = 0 \quad and \quad \left.\frac{d\,x_2(t)}{dt}\right|_{t=0} = 0 \;,$$

find the specific solutions of $x_1(t)$ *and* $x_2(t)$. Plot both $x_1(t)$ *and* $x_2(t)$ for $t = 0$ *to* 5 and show them in a single display. **(9 marks)**

Question three

(a) Given a scalar function $f = 3x^2 y - 7yz^2 + xyz$,

(i) find the value of $\vec{\nabla} f$ at the point P : (1 , 2 , 3) , **(3 marks)**

(ii) find the directional derivative of f at the point P : (1 , 2 , 3) along the direction of [- 2 , 4 , 5] . **(4 marks)**

(b) Given a vector field $F = [\, 15x^2 y + 4x^3 ,\, 5x^3 ,\, -12z^3 \,]$, find the value of the line integral of F from the point P_1 : (1 , 10 , 0) to the point P_2 : (2 , 5 , 0) along a line path of L , i.e., $\int_{P_1, L}^{P_2} \vec{F} \bullet d\vec{l}$,

(i) if L : $z = 0$ and $y = -5x + 15$, **(7 marks)**

(ii) if L : $z = 0$ and $y = \frac{10}{x}$, **(7 marks)**

(iii) is the given F a conservative vector field ? If so, then find its associated scalar potential. **(4 marks)**

Question four

(a) Given a vector field $F = [\, 7\,x\,y\,,\, -3\,z^2\,,\, 2\,x\,z\,]$, find the value of the surface integral of F , i.e., $\iint_S \vec{F} \bullet d\vec{s}$, if

$S : x^2 + y^2 = 25 \;,\; x > 0 \;,\; y > 0$ and $-1 \le z \le +4$. **(10 marks)**

(b) Given a vector field $G = [\, 5\,x^3\,y\,,\, 6\,y\,z^3\,,\, -3\,\sinh(3\,z)\,]$, utilize the divergence theorem to find the closed surface integral of G , i.e., $\oiint_S \vec{G} \bullet d\vec{s}$ if

S is the closed surface enclosing the following volume

$V : \; 1 \le x \le 5 \;,\; 2 \le y \le 8$ and $-1 \le z \le 6$. **(10 marks)**

(c) For any vector field $F = [\, F_x(x,y,z)\,,\, F_y(x,y,z)\,,\, F_z(x,y,z)\,]$, show that

$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F} \right) \equiv 0$. **(5 marks)**

Question five

Given the following one-dimensional wave equation for an elastic string of length L as

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

(a) set $u(x,t) = F(x)\, G(t)$ and utilize the separation of variable scheme to break the above partitial differential equation into two ordinary differential equation,

(6 marks)

(b) The general solution of the given partial differential equation can be written as

$$u(x,t) = \sum_{\forall k} u_k(x,t)$$
$$= \sum_{\forall k} \left(A_k \cos(k\,x) + B_k \sin(k\,x)\right)\left(C_k \cos(c\,k\,x) + D_k \sin(c\,k\,x)\right)$$

where A_k , B_k , C_k & D_k *are arbitrary constants*

(i) applying two fixed end conditions (i.e., $u_k(0,t) = 0 = u_k(L,t)$) and zero initial speed condition (i.e., $\left.\frac{\partial u_k(x,t)}{\partial t}\right|_{t=0} = 0$), deduce from the above general solution that $u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{c\,n\pi t}{L}\right)$

where E_n (n = 1 , 2 , 3 , ...) are arbitrary constants. **(8 marks)**

(ii) if c = 5 , L = 9 and the initial position of the string is given as

$$u(x,0) = \begin{cases} 2\,x & if \quad 0 \le x \le 3 \\ -x+9 & if \quad 3 \le x \le 9 \end{cases}$$

find the values of E_1 , E_2 , E_3 , $\cdots\cdots$, E_{10} . Write down the specific solution of u(x,t) in its series expression up to E_{10} term.

(11 marks)