UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

MAIN EXAMINATION 2008/2009

TITLE OF PAPER	:	LINEAR ALGEBRA AND VECTOR CALCULUS
COURSE NUMBER	:	E372
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN. STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

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<u>**E372 Linear Algebra and Vector Calculus**</u>

Question one

(a) Given the following matrix equation $AX = b$ where

$$A = \begin{pmatrix} 0 & 3 & -2 \\ 1 & -2 & 6 \\ 4 & 5 & -1 \end{pmatrix} , \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad and \quad b = \begin{pmatrix} -21 \\ 29 \\ -24 \end{pmatrix}$$

use the Gauss elimination method to find the solution of X. **(7 marks)**

(b) Given the following differential equations for a coupled oscillator system as

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -5\, x_1(t) + 16\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 4\, x_1(t) - 17\, x_2(t) \end{cases}$$

(i) find the eigen frequencies ω and their respective eigen vectors of X ,

(6 marks)

(ii) write down the general solutions of $x_1(t)$ *and* $x_2(t)$ in terms of the eigen frequencies and eigen vectors obtained in (b)(i) , **(3 marks)**

(ii) if initial conditions are given as

$$x_1(0) = -4 \ , \quad x_2(0) = 1 \ , \quad \left.\frac{d\,x_1(t)}{dt}\right|_{t=0} = -2 \quad and \quad \left.\frac{d\,x_2(t)}{dt}\right|_{t=0} = 3 \ ,$$

find the specific solutions of $x_1(t)$ *and* $x_2(t)$. Plot these specific solutions for $t = 0$ *to* 5 and show them in a single display.

(9 marks)

Question two

(a) Given any vector function $\vec{F} = \vec{e}_x\ F_x(x,y,z) + \vec{e}_y\ F_y(x,y,z) + \vec{e}_z\ F_z(x,y,z)$,

show the following vector identity that $\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$. **(5 marks)**

(b) Given a vector field $\vec{F} = -\vec{e}_x\ 3\,y^2\,e^{-3x} + \vec{e}_y\left(2\,y\,e^{-3x} - 7\,z^2\right) - \vec{e}_z\ 14\,y\,z$, find the value of line integral of $\vec{F}$ from the point $P_1 : (1, 2, 0)$ to the point $P_2 : (7, 10, 0)$ along a line path of L , i.e., $\int_{P_1, L}^{P_2} \vec{F} \bullet d\vec{l}$,

(i) if L is a straight line from P_1 to P_2 , **(7 marks)**

(ii) if L is a semicircular path from P_1 to P_2 in counter clockwise sense , i.e., with a radius of 5 and centred at (4 , 6 , 0)

(10 marks)

(iii) is the given $\vec{F}$ a conservative vector field ? If so, then find its associated scalar potential. **(3 marks)**

Question three

Given the following differential equation as :

$$2\frac{d^2 y(x)}{d x^2} + 3\frac{d\, y(x)}{d x} + 6\, y(x) = 0$$

utilize the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$,

(a) write down the indicial equations. Find the values of s and a_1 (by setting $a_0 = 1$) . **(7 marks)**

(b) write down the recurrence relation. For all the appropriate values of s and a_1 in (a), set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_8 . Thus write down two independent solution in their polynomial forms. Also write down the general solution of the given differential equation, **(10 marks)**

(c) if the initial conditions are $y(0) = -2$ & $\left.\frac{d\, y(x)}{d x}\right|_{x=0} = 1$, determine the values of the arbitrary constants of the general solution in (b). Then plot this specific solution of $y(x)$ for $x = 0$ *to* 3 . **(8 marks)**

Question four

(a) Given the following partial differential equation

$$x^2\, y \frac{\partial^2 u(x,y)}{\partial x^2} + x^2\, y \frac{\partial u(x,y)}{\partial x} = -\, x\, y^2 \frac{\partial^2 u(x,y)}{\partial y^2}$$

set $u(x,y) = F(x)\, G(y)$ and utilize the separation of variable scheme to break the above partitial differential equation into two ordinary differential equation.

(8 marks)

(b) The general solution of a one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$ can be written as

$$u(x,t) = \sum_{\forall k} u_k(x,t)$$
$$= \sum_{\forall k} \left(A_k \cos(k\,x) + B_k \sin(k\,x)\right)\left(C_k \cos(c\,k\,t) + D_k \sin(c\,k\,t)\right)$$

$where \quad A_k\ , B_k\ ,C_k\ \&\ D_k \quad are\ arbitrary\ constants$

(i) by direct substitution, show that the above $u_k(x,t)$ satisfies the given wave equation, **(4 marks)**

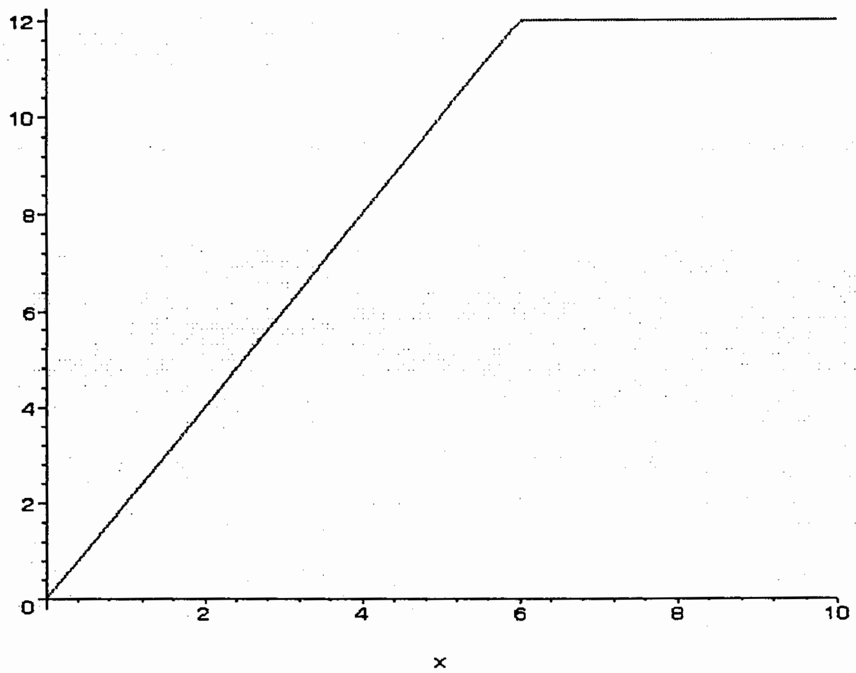
(ii) after applying two fixed end conditions and one zero initial speed condition , the above general solution can be deduced to $u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{c n \pi t}{L}\right)$ where E_n (n = 1 , 2 , 3 , ...) are arbitrary constants. If c = 2 , L = 6 and the initial position of the string is given as $u(x,0) = \begin{cases} 5\,x & if \quad 0 \le x \le 1 \\ -\,x + 6 & if \quad 1 \le x \le 6 \end{cases}$

find the values of $E_1\ , E_2\ , E_3\ , \cdots\cdots , E_8$. Then plot this specific polynomial solutions of t = 0 , t = 0.1 and t = 0.2 all for the same range of $x = 0 \quad to \quad 6$ and show them in a single display.

(13 marks)

Question five

(a) Given a periodic function $f(x)$ plotted for one period (x = 0 to 10) as



where the bending point of $f(x)$ occurs at $x = 6$ & $f(6) = 12$,

(i) express the above $f(x)$ in terms of step functions and reproduce the above diagram, **(4 marks)**

(ii) find the Fourier series of $f(x)$ truncated after $n = 4$ (i.e., the first five partial sums of its cosine series plus the first four partial sums of its sine series) , **(8 marks)**

(iii) plot the truncated Fourier series in (ii) for x = 0 to 30 . **(5 marks)**

(b) Given the following non-periodic function $g(x)$ as

$$g(x) = \begin{cases} 0 & if \quad x < 0 \\ 6\,e^{-2x} & if \quad x > 0 \end{cases}$$

express $g(x)$ in terms of its Fourier integral. **(8 marks)**