UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRONIC ENGINEERING

MAIN EXAMINATION 2009/2010

TITLE OF PAPER : LINEAR ALGEBRA AND VECTOR CALCULUS

COURSE NUMBER : E372

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

THIS PAPER HAS SIX PAGES, INCLUDING THIS PAGE.

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E372 Linear Algebra and Vector Calculus

Question one

(a) Given the following system of linear equations as :

$$\begin{cases} -10\,x_1 + 5\,x_2 - 8\,x_3 = 41 \\ 7\,x_1 - 3\,x_2 + 6\,x_3 = -28 \\ 6\,x_1 - 3\,x_2 + x_3 = -17 \end{cases}$$

(i) solve them by Gauss elimination , **(4 marks)**

(ii) solve them by Crammer's rule . **(4 marks)**

(b) Given the following system of first order differential equations as :

$$\begin{cases} \dfrac{d\,x_1(t)}{d\,t} = 9\,x_1(t) - 3\,x_2(t) \\ \dfrac{d\,x_2(t)}{d\,t} = 4\,x_1(t) - 4\,x_2(t) \end{cases}$$

(i) Set $x_1(t) = X_1\,e^{\lambda t}$ & $x_2(t) = X_2\,e^{\lambda t}$ and deduce the following matrix equation $A\,X = \lambda\,X$, where $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. **(4 marks)**

(ii) Find the eigenvalues λ . For each eigenvalue find its eigenvector. **(4 marks)**

(iii) Write down the general solutions of $x_1(t)$ & $x_2(t)$. **(2 marks)**

(iv) If the following initial conditions are given as $x_1(0) = 3$ & $x_2(0) = -2$, find the specific solutions of $x_1(t)$ & $x_2(t)$. Plot these $x_1(t)$ & $x_2(t)$ for t from 0 to 1 and show them in a single display . **(7 marks)**

Question two

(a) Given a scalar function as $f(x,y,z) = x^2\, z - 5\, y^3 + 4\, x\, z^2$,

(i) find the value of $\vec{\nabla} f$ at the point (- 1 , - 3 , 5), **(3 marks)**

(ii) find the value of its directional derivative , i.e., $\dfrac{df}{dl}$, at the given point (- 1 , - 3 , 5) along the direction of [2 , 1 , - 3]. **(4 marks)**

(b) Given a vector field as $\vec{F} = \vec{e}_x \left(3\, y^2 - 12\, x\, z\right) + \vec{e}_y\, 6\, x\, y - \vec{e}_z\, 6\, x^2$, find the value of $\int_{P_1,L}^{P_2} \vec{F} \bullet d\vec{l}$ where $P_1 : (1,2,0)$ & $P_2 : (7,10,0)$ and if

(i) L : a straight line from P_1 to P_2 on z = 0 plane , **(6 marks)**

(ii) L : a semi-circular path from P_1 to P_2 in counter clockwise sense on z = 0 plane.
Compare this answer with that obtained in (b)(i) and comment on the conservative property of the given vector field.
(Hint : radius = 5 & centered at (4 , 6) , thus
$x = 4 + 5\cos(t)$ & $y = 6 + 5\sin(t)$ where t is integrated
from $\pi + \tan^{-1}\left(\dfrac{4}{3}\right)$ to $2\,\pi + \tan^{-1}\left(\dfrac{4}{3}\right)$). **(6 marks)**

(iii) Find $\vec{\nabla} \times \vec{F}$. Does it agree with your comment in (b)(ii)? **(3 marks)**

(iv) If $\vec{\nabla} \times \vec{F} = 0$ in (b)(iii) , then find the associated scalar potential of the given $\vec{F}$. **(3 marks)**

Question three

Given a vector field as $\vec{F} = \vec{e}_x\, 2\, x\, z + \vec{e}_y\, 5\, x\, y + \vec{e}_z\, y^2$,

(a) find the value of $\int_S \vec{F} \bullet d\vec{s}$ if the surface S is given as :

$S: \ 4x^2 + y^2 = 4 \quad , \quad 1 \le z \le 5$

(Hint : set $x = \cos(t)$ & $y = 2\sin(t)$ *where* $0 \le t \le 2\pi$) **(10 marks)**

(b) utilize the Divergence theorem , i.e., $\oiint_S \vec{F} \bullet d\vec{s} \equiv \iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$, and find the value of $\oiint_S \vec{F} \bullet d\vec{s}$ if the closed surface S is the cover surface of a box with $0 \le x \le 1 , 0 \le y \le 2 \ \& \ 0 \le z \le 3$, **(7 marks)**

(c) use the given $\vec{F}$ to show that it satisfies the following vector identity :

$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \vec{e}_x \left(\nabla^2 F_x\right) - \vec{e}_y \left(\nabla^2 F_y\right) - \vec{e}_z \left(\nabla^2 F_z\right)$. **(8 marks)**

Question four

Given the following non-homogeneous differential equation as :

$$\frac{d^2 y(t)}{dt^2} - 3\frac{d\,y(t)}{dt} + 2\,y(t) = f(t)$$

where $f(t)$ is a periodic function with its period = 2 , i.e.,

$f(t) = f(t+2) = f(t+4) = f(t+6) = \cdots\cdots$, and its first period behaviour is given as

$$f(t) = \begin{cases} t & if \quad 0 \le t \le 1 \\ -t+2 & if \quad 1 \le t \le 2 \end{cases},$$

(a) (i) find the Fourier series representation of $f(t)$ up to $n = 10$ and name this truncated series as $f_{10}(t)$, **(7 marks)**

(ii) find the particular solution of $y(t)$ corresponding to $f_{10}(t)$ replacing $f(t)$ in the given non-homogeneous differential equation , **(9 marks)**

(b) (i) find the general solution for the homogeneous part of the given differential equation , i.e., $\frac{d^2 y(t)}{dt^2} - 3\frac{d\,y(t)}{dt} + 2\,y(t) = 0$, then write down the general solution for the given non-homogeneous differential equation , **(4 marks)**

(ii) find the specific solution to the given non-homogeneous differential equation if the initial conditions are given as

$$y(0) = -5 \quad \& \quad \left.\frac{d\,y(t)}{dt}\right|_{t=0} = 2 \quad .$$ **(5 marks)**

Question five

A vibrating string of length L is fixed at its two ends , i.e., x = 0 & x = L . Its transverse displacement $u(x,t)$ satisfies the following one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$ where c is a constant related to the properties of the given string ,

(a) set $u(x,y) = F(x)\,G(y)$ and utilize the separation of variable scheme to break the above partitial differential equation into two ordinary differential equations.

(4 marks)

(b) The general solution of the above partitial differential equation can be written as

$$u(x,t) = \sum_{\forall k} u_k(x,t)$$

$$= \sum_{\forall k} \left(A_k \cos(k\,x) + B_k \sin(k\,x)\right)\left(C_k \cos(c\,k\,t) + D_k \sin(c\,k\,t)\right)$$

where A_k , B_k , C_k & D_k are arbitrary constants.

(i) Applying two fixed end conditions, i.e., $u_k(0,t) = 0$ & $u_k(L,t) = 0$ and one zero initial speed condition , i.e., $\left.\frac{\partial u_k(x,t)}{\partial t}\right|_{t=0} = 0$, show that the above general solution can be deduced to

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{c\,n\pi t}{L}\right)$$ where E_n (n = 1 , 2 , 3 , ...) are arbitrary constants.

(8 marks)

(ii) If c = 3 , L = 10 and the initial position of the string is given as

$$u(x,0) = \begin{cases} 3x & \text{if} \quad 0 \le x \le 2 \\ 6 & \text{if} \quad 2 \le x \le 7 \\ -x+10 & \text{if} \quad 7 \le x \le 10 \end{cases}$$

find the values of E_1 , E_2 , E_3 , ······ , E_6 . Then plot this specific polynomial solutions of t = 0 , t = 0.3 and t = 0.6 all for the same range of $x = 0$ *to* 10 and show them in a single display.

(13 marks)