UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

MAIN EXAMINATION 2009/2010

TITLE OF PAPER : COMPLEX VARIABLES

COURSE NUMBER : E471

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

THIS PAPER HAS SIX PAGES, INCLUDING THIS PAGE.

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E471 Complex Variables

Question one

(a) Given the following function of (x, y) as

$u(x,y) = (x^3 - 3xy^2)e^{(-x^2+y^2)}\cos(axy) + (3x^2y - y^3)e^{(-x^2+y^2)}\sin(axy)$,

(i) determine the value of **a** such that the given u(x,y) is a harmonic function, **(3 marks)**

(ii) then find its conjugate harmonic function v(x,y) . **(4 marks)**

(b) Given $w = \dfrac{z-3}{3z-1}$,

(i) plot its mapped line in w-plane from a unit circle, centered at the origin and having a radius of 1 , in z-plane and show that the mapped line is also a unit circle. **(3 marks)**

(ii) show that the right most point of the unit circle in z-plane (i.e., x = 1 , y = 0) can be mapped to the left most point of the unit circle in w-plane (i.e., u = - 1 , v = 0). Also show that the left most point of the unit circle in z-plane (i.e., x = -1 , y = 0) can be mapped to the right most point of the unit circle in w-plane (i.e., u = + 1 , v = 0). **(3 marks)**

(iii) inverse the given w and express z in terms of w , then plot its mapped line in z-plane from a circle, centered at the origin and having a radius of 0.5 in w-plane . If the mapped line in z-plane is a circle, then write down the centre of the circle as well as the radius of the circle. **(4 marks)**

(c) Given $w = \cos(z)$,

(i) plot its mapped line in w-plane from a quarter of a circle, centered at the origin and having a radius of 10 , in the third quadrant in z-plane, **(4 marks)**

(ii) plot its mapped line in w-plane from a line segment of x = 1 and $0 < y < 4$ in z-plane. **(4 marks)**

Question two

(a) Given $f(z)=\dfrac{e^{5z}}{z-2}$ and $P_1:(-1,-6)$ & $P_2:(4,9)$, find the value of

$\int_{P_1}^{P_2} f(z)\,dz$ if the integration path is :

(i) a straight line joining P_1 & P_2 . Name the answer as A1.

(6 marks)

(ii) a parabolic path joining P_1 & P_2 with $y=x^2-7$. Name the answer as A2.

(6 marks)

(iii) show that $A2-A1=2\pi i\left(e^{5z}\right)_{z=2}$ and make a brief comment.

(4 marks)

(b) Given $f(z)=\dfrac{4}{z^2-2z+5}$, find its Laurent series of $f(z)$ about the expansion centre $z_0=3+5i$ and define its convergent range. **(9 marks)**

Question three

(a) Given $\int_0^{2\pi} \frac{1 + 3\sin(4\theta)}{\cos(2\theta) + 3} \, d\theta$,

(i) use int command to find its answer, **(2 marks)**

(ii) convert it into a complex contour integral and find its answer. Compare this answer with that obtained in (a)(i). **(7 marks)**

(b) Convert the following definite integrals into complex contour integrals and utilize the residue theorem to find

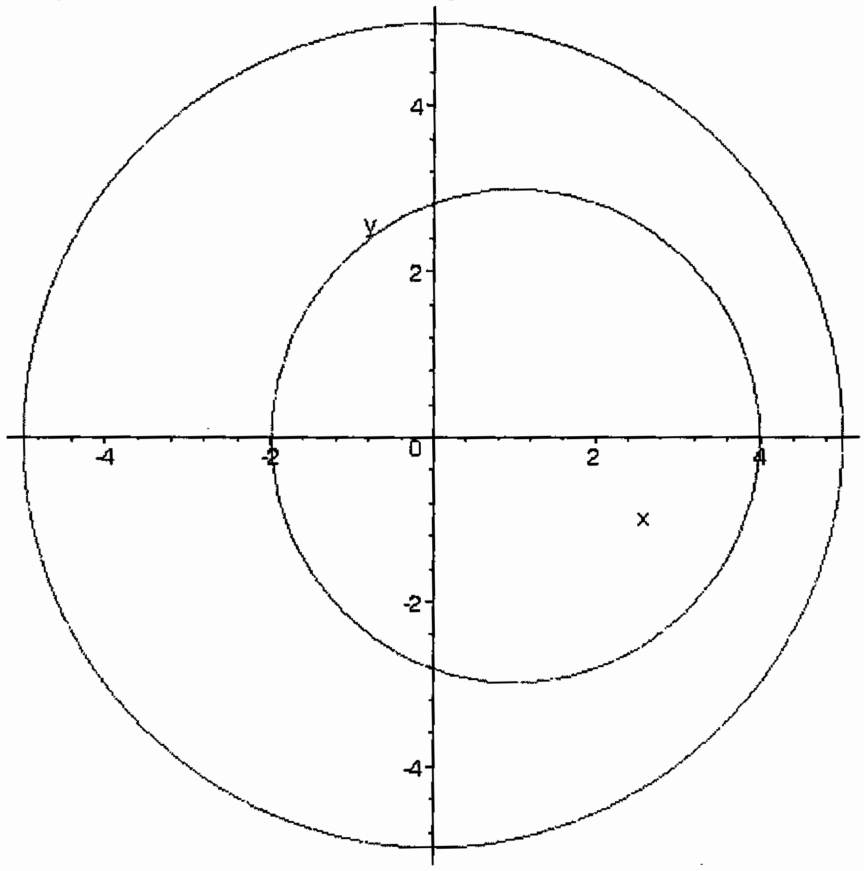
(i) the value of $\int_{-\infty}^{\infty} \frac{1}{x^4 - 2x^3 + 9x^2 - 68x + 260} \, dx$ **(6 marks)**

(ii) the values of $\int_{-\infty}^{\infty} \frac{\cos(5x)}{x^2 + x + 4} \, dx$ and $\int_{-\infty}^{\infty} \frac{\sin(5x)}{x^2 + x + 4} \, dx$

(10 marks)

Question four

Given a very long two non-coaxial cable system as shown below



The outer hollow cable is having a radius of 5 and centred at the origin while the inner cable is having a radius of 3 and centred at x = 1 & y = 0 .
If the potential on the outer cable is 20 volt and the inner cable is zero volts, use the potential theorem to find the solution of the potential f(x,y) in the region between two cables as outlined in the following steps :

(a) scale down the outer radius to the length of unity and find the appropriate value of b in $w = \dfrac{z-b}{b\,z-1}$ such that it can map these two off-centered cylinders in the z-plane to a pair of coaxial cylinders in the w-plane. Find also the radius of the mapped inner cylinder in the w-plane. **(10 marks)**

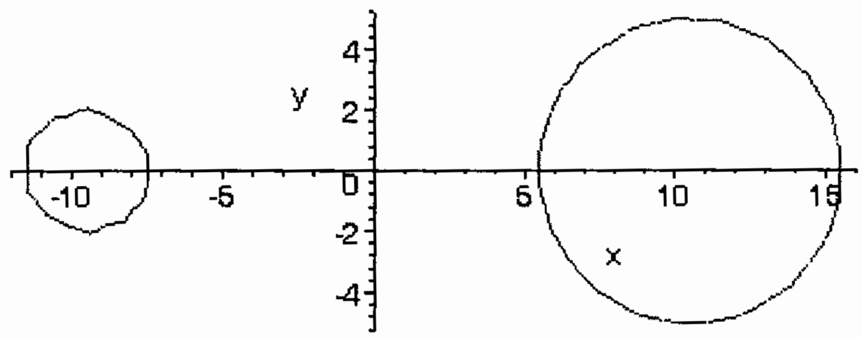
(b) (i) write down its general solution to the two dimensional Laplace equation , i.e., potential function f , in terms of w , **(2 marks)**

(ii) use the above given boundary conditions to deduce a specific potential function f(x,y) , **(6 marks)**

(iii) plot the equal potential surfaces of 0 , 5 , 10 , 15 and 20 volts in the z-plane and show them in a single display. **(7 marks)**

Question five

For a very long two parallel transmission lines system, if the radius of two lines are 2 and 5 respectively with the smaller size line on the left and their central axis are separated by a distance of 20 as shown in the diagram below



if these two lines are maintained at a potential difference of 3 (zero potential on the left line and 3 volts on the right line), use the potential theorem to find the solution of the potential f(x,y) in the region between two lines as outlined in the following steps :

(a) Choose the appropriate origin, i.e., the left line is centred at $(x = -h_1 , y = 0)$ and the right line is centred at $(x = +h_2 , y = 0)$, use $w = \dfrac{z+c}{z-c}$ and find the appropriate values of c , h_1 & h_2 such that it can map these two transmission line in the z-plane into a pair of coaxial cables in the w-plane. Find also the radius of these coaxial cables. **(14 marks)**

(b) (i) write down its general solution to the two dimensional Laplace equation , i.e., potential function f , in terms of w , **(2 marks)**

(iii) use the above given boundary conditions to deduce a specific potential function f(x,y) , **(4 marks)**

(iii) plot the equal potential surfaces of 0 , 1 , 2 and 3 volts in the z-plane and show them in a single display. **(5 marks)**