UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

MAIN EXAMINATION 2010/2011

TITLE OF PAPER : COMPLEX VARIABLES

COURSE NUMBER : E471

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

THIS PAPER HAS EIGHT PAGES, INCLUDING THIS PAGE.

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Question one

(a) Given $u(x,y) = 2x^3 - 6xy^2 - 4e^{-x}\cos(y)$,

(i) show that $u(x,y)$ is a harmonic function , **(3 marks)**

(ii) find its conjugate harmonic function , $v(x,y)$. **(5 marks)**

(b) Given $f(z) = \dfrac{\ln(2z)}{z-1}$, $z_1 = -1 - i$ and $z_2 = 7 + 7i$,

find the value of $\int_{z_1, L}^{z_2} f(z)\,dz$

(i) if L is a straight line from z_1 to z_2 , **(8 marks)**

(ii) if L is a parabolic line path from z_1 to z_2 described by

$y = \dfrac{1}{6}x^2 - \dfrac{7}{6}$. Compare this result with that obtained in (b)(i) and make brief comment. **(9 marks)**

Question two

(a) Given $f(z) = \dfrac{4i}{z^2 - 2z + 5}$,

(i) convert the given $f(z)$ into its partial fraction, **(3 marks)**

(ii) given the expansion centre as $z_0 = i$, find its Laurent series and specify the region such that this Laurent series is convergent.

(9 marks)

(b) Given a definite integral of $\int_0^{2\pi} \dfrac{5 - \cos(3\theta)}{18 - 13\sin(4\theta)}\, d\theta$,

(i) use int command to find its value, **(2 marks)**

(ii) convert it to a complex contour integral and utilize the residue theorem to find its value. Compare this value with that obtained in (b)(i) and make a brief comment. **(11 marks)**

Question three

Convert the following definite integrals into complex contour integrals and utilize the residue theorem to find

(a) the value of $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 6x^2 + x + 12}$ **(6 marks)**

(b) the values of $\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2 - 2x + 4} dx$ and $\int_{-\infty}^{\infty} \frac{\sin(3x)}{x^2 - 2x + 4} dx$ **(10 marks)**

(c) the principal value of $\int_{-\infty}^{\infty} \frac{2x - 5}{x^3 - 4x^2 + 7x - 12} dx$ **(9 marks)**

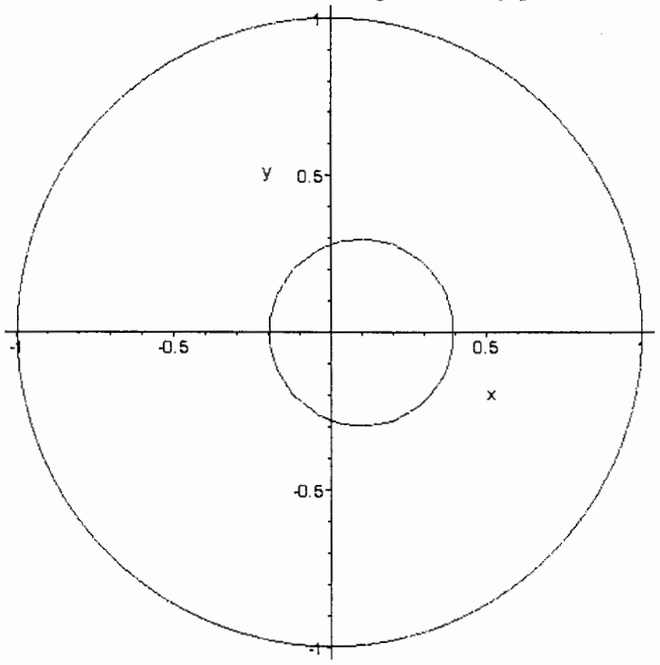
Question four

(a) Find the linear fractional transformation $w = \dfrac{a\,z + b}{c\,z + d}$ such that it maps

$$\left(z_1 = -1 \;\; to \;\; w_1 = -\frac{3}{2}\right) \;\; , \;\; \left(z_2 = 0 \;\; to \;\; w_2 = -\frac{2}{3}\right) \;\; \& \;\; \left(z_3 = i \;\; to \;\; w_1 = -\frac{1}{2} + \frac{1}{2}i\right)$$

(6 marks)

(b) Two long hollow metal cylinders eccentrically located one inside the other have their circular cross sections lying on the x-y plane as shown below :



The outer cylinder has a radius of unity and centered at the origin while the inner one has a radius of 0.3 and centered at $(0.1\,,\,0)$.

(i) Find the appropriate value of b in $w = \dfrac{z - b}{b\,z - 1}$ such that it can map these two off-centered cylinders in the z-plane to a pair of coaxial cylinders in the w-plane. Find also the radius of the mapped inner cylinder in the w-plane. **(8 marks)**

(ii) If the electric potential of the outer cylinder is 6 volts and that of the inner one is zero, find the potential between the cylinders. Plot the equal potential surfaces of $0, 2, 4$ and 6 volts in the z-plane and show them in a single display. **(11 marks)**

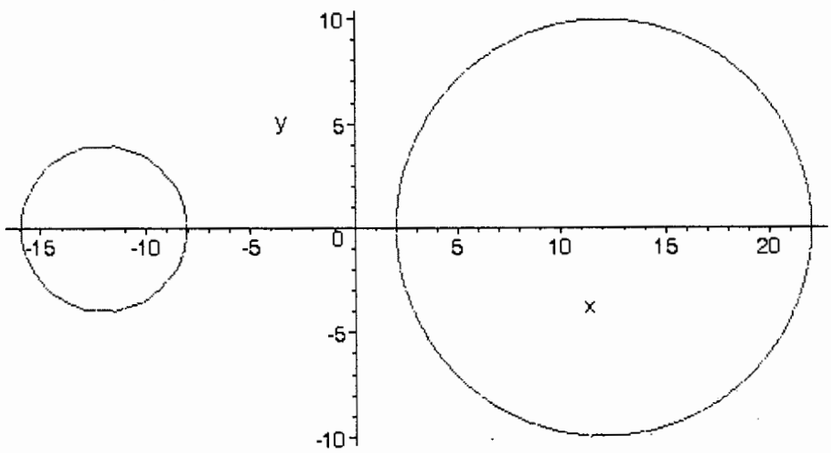
Question five

(a) Given a linear fractional transformation $w = \dfrac{z+3}{z-3}$,

(i) find its inverse transformation , **(2 marks)**

(ii) use *conformal* command to plot both $|w| = 0.25$ and $|w| = 4$ circles in $w-$ plane onto their mapping in the z – plane. Show them in a single display. **(6 marks)**

(b) Two long parallel transmission lines with the cross-section radius of the left and right lines as 4 and 10 respectively, and a separation between their central axis of 24 , with their circular cross sections lying on the x-y plane as shown below :



(i) Find the appropriate value of c given in $w = \dfrac{z+c}{z-c}$ such that it can map these two transmission line in the z-plane (with certain shift of the origin along the real axis as given in the above diagram) into a pair of coaxial cables in the w-plane. Find also the radius of these coaxial cables. **(7 marks)**

(ii) If the electric potential of the left line is zero and that of the right one is 9, find the potential between the transmission lines. Plot the equal potential surfaces of 0, 3, 6 and 9 volts in the z-plane and show them in a single display. **(10 marks)**