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UNIVERSITY OF SWAZILAND

Final Examination 2005

Title of Paper : Linear Algebra

Program : BSc./B.Ed./B.A.S.S. II

Course Number : M 220

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of SEVEN questions on THREE pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

Special Requirements: NONE

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

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Question 1

(a) Find all values of b_1, b_2 and b_3 such that the linear system

$$\begin{cases} x_1 + 3x_2 - x_3 = b_1 \\ x_1 - x_2 + 2x_3 = b_2 \\ 4x_2 - 3x_3 = b_3 \end{cases}$$

is consistent. Solve the system in the consistent case.

[10 marks]

(b) Determine whether the subset $\{[2y+z, y, z] \mid y, z \in \Re\}$ is a subspace of $\Re^3$.

[5 marks]

(c) Let v_1, v_2, v_3 be a set of linearly independent vectors. Prove that the vectors $w_1 = 3v_1$, $w_2 = 2v_1 - v_2$, $w_3 = v_1 + v_3$ are also linearly independent.

[5 marks]

Question 2

(a) Find a basis for the nullspace of the matrix: $\begin{bmatrix} 1 & -2 & 1 & 1 \\ 2 & 1 & -3 & -1 \\ 1 & -7 & -6 & 2 \end{bmatrix}$.

[10 marks]

(b) Find a matrix C such that: $C\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -6 \end{bmatrix}$.

[10 marks]

Question 3

(a) Using the two linearly independent vectors in $\Re^4$, $[2, 1, 1, 1]$ and $[1, 0, 1, 1]$, form a basis for $\Re^4$.

[10 marks]

(b) Find a basis for the subspace of P, the space of all polynomials, spanned by $x^2 - 1, x^2 + 1, 4, 2x - 3$.

[10 marks]

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Question 4

(a) Find all real eigenvalues and corresponding eigenvectors of the matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}.$$

[10 marks]

(b) Find the unit vector in the same direction as $v = \begin{bmatrix} 4 \\ -12 \\ 3 \end{bmatrix}$. **[5 marks]**

(c) Determine whether the map $T': \Re^3 \to \Re^2$ defined by:

$$T([x_1, x_2, x_3]) = [2x_1 + x_2 + x_3,\ x_1 + x_2 + 3x_3]$$

is a linear transformation.

[5 marks]

Question 5

(a) If $T: \Re^2 \to \Re^3$ is defined by $T([x_1, x_2]) = [2x_1 + x_2,\ x_1,\ x_1 - x_2]$ and $T': \Re^3 \to \Re^2$ is defined by $T'([x_1, x_2, x_3]) = [x_1 - x_2 + x_3,\ x_1 + x_2]$, find the standard matrix representation for the linear transformation $T' \circ T$ that carries $\Re^2 \to \Re^2$. Find a formula for $T' \circ T([x_1, x_2])$.

[10 marks]

(b) Determine whether the set $\Re^2$ with the usual addition but with scalar multiplication defined by $r[x, y] = [ry, rx]$ is a vector space under these operations.

[10 marks]

Question 6

(a) Let V be a vector space with basis $\{v_1, v_2, v_3, \ldots, v_n\}$, and let $W = sp(v_3, v_4, \ldots, v_n)$. If $w = r_1 v_1 + r_2 v_2$ is in W, show that $w = 0$.

[5 marks]

(b) Find a basis for the row space, a basis for the column space, a basis for the null space, the rank and nullity of the following matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}.$$

Hence verify the formula rank(A) + nullity(A) = n.

[15 marks]

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Question 7

(a) Let W be the subspace of $\Re^4$ spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 4 \\ 3 \\ 8 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 5 \end{bmatrix}.$$

Construct a basis for W from these vectors, and hence find $\dim(W)$.

[10 marks]

(b) Given the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, find A^{-1}, and hence write A as a product of elementary matrices.

[10 marks]

****** END OF EXAMINATION ******