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# UNIVERSITY OF SWAZILAND

## SUPPLEMENTARY EXAMINATIONS 2005

## B.Sc. / B.Ed. / B.A.S.S. II

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| TITLE OF PAPER | : | FOUNDATIONS OF MATHEMATICS |
| COURSE NUMBER | : | M231 |
| TIME ALLOWED | : | THREE (3) HOURS |
| INSTRUCTIONS | : | 1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.<br>2. ANSWER ANY FIVE QUESTIONS |
| SPECIAL REQUIREMENTS | : | NONE |

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

## QUESTION 1

(a) Write the negation of the following statement: "The real number $u$ is a *least upper bound* for a set $S$ of real numbers if and only if $u$ is an upper bound for $S$ and $\forall$ real numbers $\varepsilon > 0$, $\exists\ x \in S$ such that $x > u - \varepsilon$." [5]

(b) (i) Suppose you want to show that $A \Rightarrow B$ is false. How should you do this? What should you try to show about the truths of $A$ and $B$? [2]

(ii) Apply your answer of part (b) (ii) to show that the statement "If $x$ is a real number that satisfies $-3x^2 + 2x + 8 = 0$, then $x > 0$" is false. [3]

(c) Prove that if $A \Rightarrow B$, $B \Rightarrow C$, and $C \Rightarrow A$, then $A$ is equivalent to $B$ and $A$ is equivalent to $C$. [10]

## QUESTION 2

(a) If $f(n) = 3^{2n} + 7$, where $n$ is a natural number, show that $f(n+1) - f(n)$ is divisible by 8. Hence prove by induction that $3^{2n} + 7$ is divisible by 8. [7]

(b) Show that if $r$ is a nonzero rational number, then $r\sqrt{2}$ is irrational. [6]

(c) Using the result in part (b), or otherwise, show that $\sqrt{8}$ is irrational. [7]

## QUESTION 3

(a) Prove that if $n$ points are placed on the circumference of a circle and chords are drawn from each point to all other points in such a way that no three chords have a common point of intersection, then the circle is divided into $\binom{n}{4} + \binom{n}{2} + 1$ regions. [10]

(b) Prove, by the contrapositive method, that if no angle of a quadrilateral $RSTU$ is obtuse, then the quadrilateral $RSTU$ is a rectangle. [5]

(c) Prove that in any set of $n+1$ integers, there must be two whose difference is divisible by $n$. [5]

## QUESTION 4

(a) For which positive integers $n$ is $2^n < n!$? Prove that your answer is correct.[10]

(b) (i) Give the definition of a *bijection*. [2]

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = 8 - 2x$ for all $x \in \mathbb{R}$. Show that $f$ is a *bijection* and find $f^{-1}$. [8]

## QUESTION 5

(a) Let $x = 0.a_1a_2a_3\ldots$, where for $n = 1, 2, 3, \ldots$, the value of $a_n$ is the number 0 or 1 or 2 which is the remainder on dividing $n$ by 3. Is $x$ rational? If so, express $x$ as a fraction $\frac{m}{n}$ where $m$ and $n$ are integers with $n \neq 0$. [8]

(b) Prove that between any two different real numbers there is a rational number and an irrational number. [12]

## QUESTION 6

(a) (i) Define an equivalence relation. [2]

(ii) Show that the relation

$$\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 2\}$$

is an equivalence relation. What are the equivalence classes of $\mathcal{R}$? [12]

(b) (i) Define the composition $f \circ g$ of any two functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$. [2]

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = \sin x$ and $g(x) = x^2 + 2$ for all $x \in \mathbb{R}$. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$. [4]

## QUESTION 7

(a) Prove that the square root of a natural number is rational if and only if the natural number is a perfect square. [8]

(b) Prove that there are infinitely many prime numbers of the form $4k + 3$, where $k$ is an integer. [8]

(c) Find an integer $x$ such that $x^2 + x + 41$ is not a prime. [4]

END OF EXAMINATION