UNIVERSITY OF SWAZILAND

Final Examination 2005

Title of Paper : Numerical Analysis I

Program : BSc./B.Ed./B.A.S.S. III

Course Number : M 311

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

Special Requirements: None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

Question 1

(a) Convert the binary number $111\ldots1$ (n 1s) to its decimal equivalent.

[8 marks]

(b) Demonstrate how you would reformulate the following problem in order to avoid loss of significant figures:

$$\frac{1-\cos x}{x^2}; \qquad x \approx 0.$$

[6 marks]

(c) By considering the Taylor series approximation for $f(x) = \tan^{-1} x$, formulate a method for evaluating π.

[6 marks]

Question 2

(a) To avoid computation of the derivative at each step of the Newton-Raphson iteration, the method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$$

is sometimes used. Determine the order of this method for sufficiently close starting value, and exhibit the corresponding asymptotic error constant.

[12 marks]

(b) Find the polynomial of degree ≤ 2 that passes through (-1,2), (0,1) and (1,3) in Newton forward-difference form.

[8 marks]

Question 3

(a) Let $a > 0$, and consider the iterative process

$$x_{n+1} = \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a} \qquad n \geq 0.$$

(i) Determine the positive fixed point of the iteration.

(ii) Find the order of the method for sufficiently close starting value, and determine the corresponding asymptotic error constant.

[10 marks]

(b) The function $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2 = (x-1)^3(x-2)$ has roots $\overline{x}_1 = 1$ and $\overline{x}_2 = 2$. Using $x_0 = 0.9$, apply the Newton-Raphson method once, and compute $|x_1 - \overline{x}_1|$.

Apply the secant method once with $x_0 = 0.9$ and $x_1 = 1.1$, and compute $|x_2 - \overline{x}_2|$. Briefly explain your results.

[10 marks]

Question 4

(a) The **positive** root of $f(x) = \alpha - \beta x^2 - x$ (with $\alpha > 0$, $\beta > 0$) is sought and the simple iteration $x_{n+1} = \alpha - \beta x_n^2$ is used. Show that convergence will occur for sufficiently close starting value x_0, provided

$$\alpha \cdot \beta < \frac{3}{4}.$$

[12 marks]

(b) Given distinct points $\{(x_i, f_i); i = 0(1)n\}$, where $a < x_0 < x_1 < \ldots < x_n < b$, we are to construct the polynomial $p_n(x)$, of degree $\leq n$, that interpolates f at x_i.

By considering the function

$$F(x) = E_n(x) - \frac{E_n(t)}{\prod_{i=0}^{n}(t - x_i)} \cdot \prod_{i=0}^{n}(x - x_i),$$

where $E_n(x) = f(x) - p_n(x)$ and $x_i \neq t \in (a, b)$, show that if $f \in C^{n+1}[a, b]$, the error of the interpolating polynomial at t is given by

$$E_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \prod_{i=0}^{n}(t - x_i);$$

$\xi \in (a, b)$. **[8 marks]**

Question 5

Suppose an approximation to $I = \int_0^{2h} f(x)\,dx$ is sought, and $f(x)$ in $[0, 2h]$ is approximated by the linear function through the TWO points $(0, f(0))$, and $(h, f(h))$.

(i) Write down the Lagrange representation of the polynomial that interpolates f at the two points $(0, f(0))$, and $(h, f(h))$.

[4 marks]

(ii) By integrating the polynomial in (i) above between 0 and $2h$, prove that the desired quadrature formula is simply

$$I \approx \tilde{I} = 2hf_1.$$

[8 marks]

(iii) Show, using Taylor series expansions about 0 and assuming $f \in C^1[0, 2h]$, that

$$I - \tilde{I} = 2h^2 f''(\xi); \qquad \xi \in (0, 2h).$$

[8 marks]

Question 6

(a) The iteration $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ will converge for sufficiently close x_0 to $s = 1$ for some values of c. Find the values of c for which this is true. For what value of c will the convergence be quadratic?

[10 marks]

(b) Use a three-point Gauss-Legendre Quadrature rule to approximate $I = \int_0^2 e^{-x^2}\,dx$.

Note:

$$\begin{cases} x_1 = -\sqrt{\frac{3}{5}} & w_1 = \frac{5}{9} \\ x_2 = 0 & w_2 = \frac{8}{9} \\ x_3 = \sqrt{\frac{3}{5}} & w_3 = \frac{5}{9} \end{cases}$$

[10 marks]

Question 7

(a) Find the polynomial of degree ≤ 2 that passes through (0,1), (-1,2) and (1,3) in Lagrange form.

[10 marks]

(b) Consider the six tabulated points $(x_1, f(x_1)$, $(x_2, f(x_2)$, $\ldots$, $(x_6, f(x_6)$. Suppose that $f(x_3)$ is perturbed by $\epsilon > 0$ so that the value is $f(x_3) + \epsilon$. Construct a difference table and show that the perturbation spreads through the table as higher differences are taken.

[10 marks]

******** *END OF EXAMINATION* ********