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UNIVERSITY OF SWAZILAND

Supplementary Examination 2005

Title of Paper : Numerical Analysis I

Program : BSc./B.Ed./B.A.S.S. III

Course Number : M 311

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

Special Requirements: None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

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Question 1

(a) Convert the binary number $(.00111001100110011\ldots)_2$, to its decimal equivalent.

[8 marks]

(b) Let x be approximated by X with an error e. Prove that the relative error, ϵ_r, in $\sqrt{X}$ is given by

$$|\epsilon_r| \approx \frac{1}{2}\left|\frac{e}{X}\right|.$$

[12 marks]

Question 2

(a) Let $a > 0$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \qquad n \geq 0$$

be an iterative method.

(i) Find the positive fixed point, s, of the scheme.

(ii) Assuming convergence to s, determine the order and corresponding asymptotic error constant for this method.

[10 marks]

(b) The function $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2 = (x-1)^3(x-2)$ has roots $\bar{x}_1 = 1$ and $\bar{x}_2 = 2$. Using $x_0 = 2.1$ and $x_0 = 0.9$, apply the Newton-Raphson method once for each x_0.

Apply the secant method once with $x_0 = 0.9$ and $x_1 = 1.1$, and state why the secant method appears to converge faster for the relevant root.

[10 marks]

Question 3

(a) For the scheme $x_{n+1} = x_n + c(x^2 - 5)$, find the range of values of c for which convergence to the positive fixed point is guaranteed.

[10 marks]

(b) Find the polynomial of degree ≤ 2 that passes through (-1,8), (0,1) and (1,2) in Newton forward-difference form.

[10 marks]

Question 4

Given $n+1$ distinct data points $\{(x_i, f_i); i = 0(1)n\}$, where $a < x_0 < x_1 < \ldots < x_n < b$, we are to construct the polynomial $p_n(x)$, of degree $\leq n$, that interpolates f at x_i.

(a) By considering the function

$$F(x) = E_n(x) - \frac{E_n(t)}{\prod_{i=0}^{n}(t - x_i)} \cdot \prod_{i=0}^{n}(x - x_i),$$

where $E_n(x) = f(x) - p_n(x)$ and $x_i \neq t \in (a,b)$, show that if $f \in C^{n+1}[a,b]$, the error of the interpolating polynomial at t is given by

$$E_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \prod_{i=0}^{n}(t - x_i);$$

$\xi \in (a,b)$.

[8 marks]

(b) Consider the case $n = 1$: p_1 interpolates f at x_0 and $x_1 = x_0 + h$. Prove that if $|f''(x)| \leq M$ for all $x \in [x_0, x_1]$,

$$|f(t) - p_1(t)| \leq \frac{h^2}{8} M.$$

[12 marks]

Hint: Use the bound of f'' and maximize the quadratic $\prod_{i=0}^{1}(t - x_i)$.

Question 5

Let $f(x) = (x - a)^m$, for some $a \in \mathbb{R}$ and $m \geq 2$.

(i) Formulate the Newton-Raphson iteration for finding the root a.

[4 marks]

(ii) Show that the order of the method is linear and produce the asymptotic error constant.

[6 marks]

Q5 (iii)/...

Q5 (iii) Apply the modified N-R method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)},$$

to this function f, show that the resultant iteration restores quadratic convergence, and exhibit the corresponding asymptotic error constant.

[10 marks]

Question 6

(a) Given the data

x	0	1	3	2	5
$f(x)$	2	1	5	6	-183

(i) construct a divided difference table.
(ii) write down the Newton form of the interpolating polynomial.
(iii) use this polynomial to approximate $f(2.5)$.

[10 marks]

(b) Evaluate

$$I = \int_0^{\pi} \cos(\sin x - 2x)\, dx$$

using the three-point Gauss-Legendre quadrature rule.

[10 marks]

Note: The third degree Legendre polynomial has zeros 0, $-\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{3}{5}}$. The corresponding weights are 8/9, 5/9 and 5/9 respectively. All angles are measured in radians.

Question 7

Let $p_2(x)$ be the quadratic polynomial interpolating $f(x)$ at $x_0 = 0$, $x_1 = h$ and $x_2 = 2h$.
(a) Write down the Lagrange representation of $p_2(x)$.

[4 marks]

(b) If $p_2(x)$ is used to derive a numerical integration rule for $I = \int_0^{3h} f(x)\,dx$, show that

$$I_2(f) = \frac{3h}{4}\left[f(0) + 3f(2h)\right].$$

[8 marks]

(c) Assuming that $f \in C^3[0, 3h]$, prove using Taylor series expansions that

$$I - I_2(f) = \frac{3}{8}h^4 f'''(\xi); \qquad \xi \in (0, 3h).$$

[8 marks]

******** *END OF EXAMINATION* ********