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UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2005

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Part of a railway line (superimposed on a rectangular coordinate system) follows the line $y = -x$ for $x \leq 0$, then turns to reach the point (3,0) following a cubic curve. Find the equation of this curve if the track is continuous, smooth, and has continuous curvature. [8]

(b) A curvilinear coordinate system (u, v, ϕ) is defined by

$$x = auv\cos\phi, \qquad y = auv\sin\phi, \qquad z = \tfrac{a}{2}(u^2 - v^2), \text{ where } u, v > 0,\ -\pi < \phi < \pi.$$

(i) Find the scale factors and the unit vectors.

(ii) Show that the coordinate system is orthogonal.

(iii) Find the line element and the volume element. [12]

QUESTION 2

(a) Let $\mathbf{u}(x, y, z) = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$ and $\mathbf{v}(x, y, z) = \dfrac{\mathbf{u}}{(x^2 + y^2)^{\frac{1}{2}}}$ be vectors in space.

(i) Compute the divergence and the curl of $\mathbf{u}$ and $\mathbf{v}$. [6]

(ii) Find the flow lines of $\mathbf{u}$ and $\mathbf{v}$. [8]

(b) Determine the directional derivative of $\phi(x, y) = \ln\sqrt{x^2 + y^2}$ at the point (1,0) in the direction of $\dfrac{2\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{5}}$. [6]

QUESTION 3

(a) Find the tangent plane and the normal line to the surface $x^2y + xyz - z^2 = 1$ at the point $P_0(1,1,3)$. [10]

(b) Give a formula $\mathbf{F} = M(x,y)\hat{\mathbf{i}} + N(x,y)\hat{\mathbf{j}}$ for the vector field in the plane with the properties that $\mathbf{F} = \mathbf{0}$ at the origin and that at any other point (a,b) in the plane, $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction, with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$. [10]

QUESTION 4

(a) By any method, find the integral of $H(x,y,z) = yz$ over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$. [7]

(b) Find the work done in moving a particle in the counterclockwise direction once around the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ if the force field is given by $\mathbf{F} = (3x - 4y)\hat{\mathbf{i}} + (4x + 2y)\hat{\mathbf{j}} - 4y^2\hat{\mathbf{k}}$. [6]

(c) Show that $y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z$ is exact and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z.$$

[7]

QUESTION 5

(a) Find out which of the fields given below are conservative. For conservative fields, find a potential function.

(i) $\mathbf{F} = (z+y)\hat{\mathbf{i}} + z\hat{\mathbf{j}} + (y+x)\hat{\mathbf{k}}$.

(ii) $\mathbf{F} = (y\sin z)\hat{\mathbf{i}} + x\sin z\hat{\mathbf{j}} + (xy\cos z)\hat{\mathbf{k}}$. [12]

(b) Integrate $f(x,y,z) = 2x - 6y^2 + 2z$ over the line segment C joining the points (1,1,1) and (2,2,2). [8]

QUESTION 6

(a) By any method, find the outward flux of the field $\mathbf{F} = (6x^2 + 2xy)\hat{\mathbf{i}} + (2y + x^2z)\hat{\mathbf{j}} + (4x^2y^3)\hat{\mathbf{k}}$ across the boundary of the region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$. [10]

(b) By any method, find the circulation of the field $\mathbf{F} = (x^2+y^2)\hat{\mathbf{i}} + (x+y)\hat{\mathbf{j}}$ around the triangle with vertices (1,0), (0,1), (-1,0) traversed in the counterclockwise direction. [10]

QUESTION 7

(a) Verify the divergence theorem for $\mathbf{F} = (2x-z)\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} - xz^2\hat{\mathbf{k}}\mathbf{i}$ taken over the region bounded by $x = 0$, $x = 3$, $y = 0$, $y = 3$, $z = 0$, $z = 3$. [10]

(b) Verify Green's theorem in the plane for

$$\oint_C [2x\mathrm{d}x - (3y - x)\mathrm{d}y],$$

where C is the closed curve (described in the positive direction) of the region bounded by the curves $y = x^2$ and $y^2 = x$.

END OF EXAMINATION