96

UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2005

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) A path of a roller coaster ride (superimposed on a rectangular coordinate system) consists of part of the parabola $y = \dfrac{x^2}{2}$ for $x \leq 0$, followed by a circular loop for $x \geq 0$. Find the equation of this loop if the track is *continuous*, *smooth*, and has a *continuous curvature*. [8]

(b) Parabolic coordinates (u, ν, ϕ) are defined by

$$x = au\nu\cos\phi, \quad y = au\nu\sin\phi, \quad z = \frac{a}{2}(u^2 - \nu^2)$$

where $u > 0$, $\nu > 0$, $-\pi < \phi < \pi$.

(i) Find the scale factors and the unit vectors of the coordinate system.

(ii) Show that the coordinate system is orthogonal.

(iii) Find the line element. [12]

QUESTION 2

(a) Show that the vector field $\mathbf{F} = (6xy+z^3)\hat{\mathbf{i}}+(3x^2-z)\hat{\mathbf{j}}+(3xz^2-y)\hat{\mathbf{k}}$ is irrotational. Find a function ϕ such that $\mathbf{F} = \nabla\phi$. [10]

(b) Let $\mathbf{u}(x, y, z) = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$ and $\mathbf{v}(x, y, z) = \dfrac{\mathbf{u}}{(x^2 + y^2)^{\frac{1}{2}}}$ be vectors in space.

(i) Compute the divergence and the curl of $\mathbf{u}$ and $\mathbf{v}$.

(ii) Find the flow lines of $\mathbf{u}$ and $\mathbf{v}$. [10]

QUESTION 3

(a) Find a vector field $\mathbf{F}(x, y, z) = M(x, y, z)\hat{\mathbf{i}} + N(x, y, z)\hat{\mathbf{j}} + P(x, y, z)\hat{\mathbf{k}}$ with the property that at each point (x, y, z) $\mathbf{F}$ points away from the origin and its magnitude $|\mathbf{F}|$ is proportional to the square of the distance from (x, y, z) to the origin. [6]

(b) Find the unit outward normal vector to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a, b > 0$$

at the point $P\left(\frac{-a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. [8]

(c) Prove that div curl $\mathbf{F} = 0$, where $\mathbf{F}$ is a twice differentiable function. [6]

QUESTION 4

(a) Let $\mathbf{F}(x, y) = (2xy - y^4 + 3)\hat{\mathbf{i}} + (x^2 - 4xy^3)\hat{\mathbf{j}}$ be a given vector field.

(i) Show that there exists a scalar potential $\phi(x, y)$ such that $\mathbf{F} = \nabla\phi$. Hence prove that the line integral $\int_C \mathbf{F}.\mathbf{dr}$ is independent of the path C.

(ii) If C is the straight line from the point (1,0) to the point (2,1), evaluate the integral given in (i). [12]

(b) Verify that the parametric equations

$$x = \rho^2 \cos\theta, \quad y = \rho^2 \sin\theta, \quad z = \rho$$

could be used to represent the surface $x^2 + y^2 - z^4 = 0$. Hence compute the unit normal to this surface at any point. [8]

QUESTION 5

Let S be the surface of the solid Ω enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$. Assuming that S is oriented outward, verify the Divergence theorem for the vector field $\mathbf{F}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ by evaluating both

$$\iint_S \mathbf{F}.\hat{\mathbf{n}}\ \mathrm{d}S \quad \text{and} \quad \iiint_\Omega \text{div}\ \mathbf{F}\ \mathrm{d}V.$$

[20]

QUESTION 6

(a) Evaluate, without using Stoke's theorem, the line integral $\int_C [xz\mathrm{d}x - y\mathrm{d}y + x^2 y\mathrm{d}z]$, where C is the edge of the base of the tetrahedron formed by $x = 0$, $y = 0$, $z = 0$, $2x + y + 2z = 8$, and the base lies on the plane $y = 0$. [8]

(b) Use Stoke's theorem to evaluate the line integral given in part (a). Hence verify Stoke's theorem. [12]

QUESTION 7

(a) Evaluate $\iint_S \mathbf{F}.\hat{\mathbf{n}}\ \mathrm{d}S$, where $\mathbf{F}(x, y, z) = -x\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + x\sin(z)\hat{\mathbf{k}}$ and S is the portion of the elliptic cylinder $\mathbf{r}(u, \nu) = 2\cos\nu\hat{\mathbf{i}} + \sin\nu\hat{\mathbf{j}} + u\hat{\mathbf{k}}$ for which $0 \leq u \leq 5$, $0 \leq \nu \leq 2\pi$. [10]

(b) By any method, find the circulation of the field $\mathbf{F} = (x^2 + y^2)\hat{\mathbf{i}} + (x + y)\hat{\mathbf{j}}$ around the triangle with vertices (1,0), (0,1), (-1,0) traversed in the counterclockwise direction. [10]

END OF EXAMINATION