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# UNIVERSITY OF SWAZILAND

## Final Examination 2005

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**Title of Paper** : Abstract Algebra I

**Program** : BSc./B.Ed./B.A.S.S. III

**Course Number** : M 323

**Time Allowed** : Three (3) Hours

**Instructions** :

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

**Special Requirements:** None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

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## Question 1

(a) Find the greatest common divisor $d$ of the numbers 52 and 20 and express it in the form $d = 52x + 20y$ for some $x, y \in \mathbb{Z}$.

**[5 marks]**

(b) Prove that every subgroup of a cyclic group is cyclic.

**[10 marks]**

(c) Give an example of a group satisfying the given conditions or if there is no such example, say so. (Do not prove anything)

(i) An abelian non-cyclic group.

(ii) A non-abelian cyclic group.

(iii) A group with no proper subgroups.

**[5 marks]**

## Question 2

(a) Suppose that $d$, $a$, $b$ are positive integers, the greatest common divisor of $a$ and $d$ equals one i.e $(a, d) = 1$ and that $d$ divides $ab$. Prove that $d$ divides $b$.

**[5 marks]**

(b) Determine all possible solutions of

$$3x \equiv 5 \pmod{11},$$

$x \in \mathbb{Z}$.

**[5 marks]**

(c) Find the number of elements in the cyclic subgroup $\langle 30 \rangle$ of $\mathbb{Z}_{42}$. (Do not list the elements)

**[5 marks]**

(d) Show that $\mathbb{R}$ under addition is isomorphic to $\mathbb{R}^+$ under multiplication.

**[5 marks]**

## Question 3

(a) Find all subgroups of $\mathbb{Z}_{18}$ and draw the lattice diagram.

**[10 marks]**

(b) Let $\varphi : G \to H$ be an isomorphism of $G$ with $H$ and let $e$ be the identity of $G$. Prove that $(e)\varphi$ is the identity in $H$ and that $(a^{-1})\varphi = [(a)\varphi]^{-1}$.

**[10 marks]**

## Question 4

(a) Prove that a non-abelian group of order $2p$, $p$ prime, contains at least one element of order $p$.

**[6 marks]**

(b) Consider the following permutations in $S_6$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

Compute (i)$\rho\sigma$ (ii) $\sigma^2$ (iii)$\sigma^{-1}$ (iv) $\sigma^{-2}$ (iv) $\rho\sigma^2$

**[10 marks]**

(c) Write the permutations in (b) as a product of disjoint cycles in $S_6$.

**[4 marks]**

## Question 5

(a) For each binary operation $*$ defined on a set $G$, say whether or not $*$ gives a group structure on the set.

(i) Define $*$ on $G = \mathbb{Q}^+$ by

$$a * b = \frac{ab}{2} \qquad \forall\, a,\, b \in \mathbb{Q}^+.$$

**[8 marks]**

(ii) Define $*$ on $G = \mathbb{R}$ by

$$a * b = ab + a + b \qquad \forall\, a,\, b \in \mathbb{R}.$$

**[8 marks]**

(b) Show that $\mathbb{Z}_6$ and $S_3$ are NOT isomorphic and that $\mathbb{Z}$ and $3\mathbb{Z}$ are isomorphic.
**[4 marks]**

## Question 6

(a) Show that $\mathbb{Z}_p$ has no proper subgroup if $p$ is prime.

**[6 marks]**

(b) Show that if $(a,\ m) = 1$ and $(b,\ m) = 1$ then

$$(ab,\ m) = 1; \qquad a,\, b,\, m \in \mathbb{Z}.$$

**[6 marks]**

(c) Prove that every group of prime order is cyclic.

**[8 marks]**

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## Question 7

(a) Let $G$ be the set of all $2 \times 2$ matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where $a, b, c \in \mathbb{Q}$ and $ac \neq 0$. Show that, with respect to matrix multiplication, $G$ is a group.

**[8 marks]**

(b) Solve the system

$$\begin{aligned} 3x &\equiv 2 (\text{mod } 5) \\ 2x &\equiv 1 (\text{mod } 3), \end{aligned}$$

$x \in \mathbb{Z}$. **[8 marks]**

(c) Prove the uniqueness of the identity element and the uniqueness of the inverse element for each element of a group $G$.

**[4 marks]**

******** *END OF EXAMINATION* ********