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# UNIVERSITY OF SWAZILAND

Supplementary Examination 2005

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**Title of Paper** : Abstract Algebra I

**Program** : BSc./B.Ed./B.A.S.S. III

**Course Number** : M 323

**Time Allowed** : Three (3) Hours

**Instructions** :

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

**Special Requirements:** None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

## Question 1

(a) Determine whether the set $\mathbb{Q}$ with respect to the binary operation

$$a * b = a + b - 2005$$

is a group.

**[8 marks]**

(b) (i) Give the definition of a cyclic group.

(ii) Prove that every finite group of prime order is cyclic.

**[8 marks]**

(c) Let $a, b, c, d, n \in \mathbb{Z}$. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

$$ac \equiv bd \pmod{n}.$$

**[4 marks]**

## Question 2

(a) Give an example of a group satisfying the given conditions, or if there no example, say so. (Do not prove anything)

(i) An abelian group

(ii) A non abelian group

(iii) A finite cyclic group

(iv) A non-abelian cyclic group

(v) A group with no proper subgroups.

**[10 marks]**

(b) Let $G$ be a group with identity $e$. Show that if $a^2 = e \; \forall \; a \in G$, then $G$ is abelian.

**[10 marks]**

## Question 3

(a) Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 8 & 1 & 5 & 4 & 3 & 7 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 7 & 3 & 8 & 6 & 1 & 2 \end{pmatrix}$$

Express $\alpha$ and $\beta$ as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even or an odd one.

**[12 marks]**

(b) Compute (i) $\alpha^{-1}$ (ii) $(\alpha\beta)^{-1}$.

**[4 marks]**

(c) Solve the equations for $x$ and $y$

(i) $\alpha x = \beta$

(ii) $y\alpha = \beta$

**[4 marks]**

## Question 4

(a) Find the number of elements in each of the cyclic groups

(i) The cyclic subgroup $\langle 25 \rangle$ of $\mathbb{Z}_{30}$.

(ii) The cyclic subgroup $\langle 30 \rangle$ of $\mathbb{Z}_{42}$.

**[4 marks]**

(b) Prove that every cyclic group is abelian.

**[8 marks]**

(c) Let $n$ be a positive integer greater than 1, and let $a, b \in \mathbb{Z}$,

$$aRb \iff a \equiv b (\text{mod } n)$$

Prove that $R$ is an equivalence relation on $\mathbb{Z}$.

**[8 marks]**

## Question 5

(a) Give the definition of a group.

**[4 marks]**

(b) Determine whether the set $G = \mathbb{Q}\setminus\{-1\}$ with respect to the binary operation

$$a * b = a + b + ab$$

is a group.

**[10 marks]**

(c) Show that $\mathbb{Z}$ and $5\mathbb{Z}$ are isomorphic and that $\mathbb{Z}_7$ and $S_3$ are NOT isomorphic.

**[6 marks]**

## Question 6

(a) Let $\varphi : G \to H$ be an isomorphism of groups.

(i) Prove that if $e_G$ and $e_H$ are identity elements of $G$ and $H$ respectively, then

$$e_H = (e_G)\varphi.$$

(ii) Prove that, $\forall\ a \in G, \quad [(a)\varphi]^{-1} = (a^{-1})\varphi.$

**[10 marks]**

(b) State Lagrange's theorem. [Do not prove]

**[4 marks]**

(c) Use the theorem in (b) to prove that a finite group of prime order has no proper subgroup.

**[6 marks]**

**Question 7**

(a) Prove that every subgroup of a cyclic group is cyclic.

**[8 marks]**

(b) Find the greatest common divisor $d$ of the number 204 and 54 and express it in the form

$$d = 204m + 54n \quad \text{for some} \quad m, n \in \mathbb{Z}.$$

**[6 marks]**

(c) Find solutions of

(i) $259x \equiv 1 \pmod{11}$

(ii) $222x \equiv 12 \pmod{18}$

**[6 marks]**

******** *END OF EXAMINATION* ********