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UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2005

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	DYNAMICS II
COURSE NUMBER	:	M 355
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Show that the function $y(x)$ which renders the functional

$$I = \int_{x_0}^{x_1} F(x, y, y')dx$$

stationary, satisfies the Euler - Lagrange equation given by

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

where $y' = dy/dx$ [6 marks]

(b) Find the curves $y(x)$ that minimize the following functionals subject to the given boundary conditions,

i.

$$I = \int_0^{\frac{\pi}{2}} (y^2 - (y')^2 - 2y \sin x)dx$$

with boundary conditions $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = 2$. [7 marks]

ii.

$$\int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right) dx$$

with boundary conditions $y(0) = \frac{1}{2}$, $y(1)$ is free [7 marks]

QUESTION 2

2. (a) Show that if f is a function of p_α, q_α and t and H is the Hamiltonian, then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]$$

[6 marks]

(b) The Hamiltonian of a two-dimensional harmonic oscillator of unit mass is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2)$$

where ω is a constant. Given that

$$A = q_1 p_2 - q_2 p_1 \quad \text{and} \quad B = \omega q_1 \sin \omega t + p_1 \cos \omega t$$

Show that both A and B are constants of motion. [8 marks]

(c) By the method of Poisson brackets, show that the transformation, given by

$$\begin{aligned} q_1 &= \sqrt{2P_1} \sin Q_1 + P_2, \qquad p_1 = \frac{1}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right) \\ q_2 &= \sqrt{2P_1} \cos Q_1 + Q_2, \qquad p_2 = -\frac{1}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right) \end{aligned}$$

is canonical. [6 marks]

<u>QUESTION 3</u>

3. A simple pendulum of mass m_2 is attached to a mass m_1 which can move freely along the horizontal line, as shown in the Figure below. The system is in a uniform gravitational field (acceleration g).

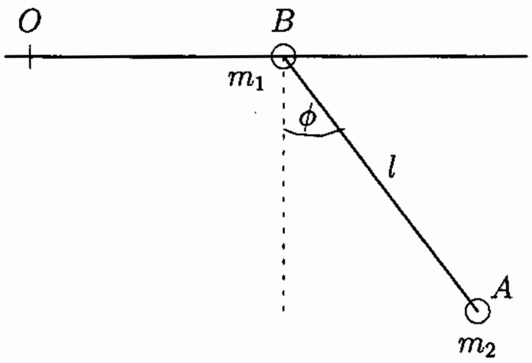

Figure 1:

Choosing the generalised coordinates to be x, the distance moved by m_1 from O, and ϕ, the inclination of BA to the vertical,

(a) Write down the transformation equation. [4 marks]

(b) Derive the Lagrangian function of the system. [6 marks]

(c) Prove that the Lagrange's equation of motion corresponding to the generalized coordinate ϕ is

$$l\ddot{\phi} + \ddot{x}\cos\phi + g\sin\phi = 0$$

[5 marks]

(d) Write down the Lagrange's equation of motion corresponding to x. [5 marks]

QUESTION 5

5. (a) Given that the transformation equations for a mechanical system are given by $\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \ldots, q_n)$, where q_α are generalized coordinates. Prove that,

(i)

$$\frac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$

(ii)

$$\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T$$

where T is the kinetic energy. [4, 4 marks]

(b) The Lagrangian for a certain dynamical system is given by

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + t^2 - 2\dot{r}t\cos\theta + 2rt\dot{\theta}\sin\theta) - mg\left(\frac{1}{2}t^2 - r\cos\theta\right) - \frac{k}{2}(r-a)^2$$

where r, θ are generalized coordinates, t is time and m, g and k are constants. Using the Lagrangian method, show that the equations of motion for the system are given by

$$\ddot{r} - r\dot{\theta}^2 = (g+1)\cos\theta - \frac{k}{m}(r-a)$$

$$\ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} + \frac{g+1}{r}\sin\theta = 0$$

[12 marks]

QUESTION 6

6. Use the following definition

$$[F, G] = \sum_{\alpha} \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} \right)$$

of a Poisson bracket between two physical quantities $F(q_\alpha, p_\alpha, t)$ and $G(q_\alpha, p_\alpha, t)$ to prove the following properties.

(a) $[u, v] = -[v, u]$ [3 marks]

(b) $[u, u] = 0$ [2 marks]

(c) $[u, v + w] = [u, v] + [u, w]$ [3 marks]

(d) $[u, vw] = v[u, w] + [u, v]w$ [3 marks]

(e) $[q_\alpha, p_\beta] = \delta_{\alpha\beta}$ [3 marks]

(f) $\dot{q}_\alpha = [q_\alpha, H]$ [3 marks]

(g) $\dot{p}_\alpha = [p_\alpha, H]$ [3 marks]

where q_α are generalized coordinates, p_α are generalized momenta, H is the Hamiltonian function and $\delta_{\alpha\beta}$ is the kronecker delta.

QUESTION 7

7. Use the Beltrami identity ($F - y'\frac{\partial F}{\partial y'}$ = Constant) to show that the extremum for the integral

$$I = \int_0^a \sqrt{\frac{1 + y'^2}{2y}} dx$$

satisfies the differential equation

$$y' = \sqrt{\frac{2c - y}{y}}.$$

By making the substitution $y = 2c\sin^2\theta$, show that the solution of the differential equation is $x = c(2\theta - \sin 2\theta)$ [20 marks]