# UNIVERSITY OF SWAZILAND

## Final Examination 2005

**Title of Paper** : Numerical Analysis II

**Program** : BSc./B.Ed./B.A.S.S. IV

**Course Number** : M 411

**Time Allowed** : Three (3) Hours

**Instructions** :

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

**Special Requirements:** NONE

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

## Question 1

(a) Determine the straight line that best fits the following data in the least squares sense: (0,1), (2,4), (5,8) and (7,13). **[10 marks]**

(b) Implement the Gram-Schmidt procedure without the normalization step so as to generate an orthogonal (NOT orthonormal) set of polynomials $\phi_0 \equiv 1$, $\phi_1(x)$, $\phi_2(x)$ on [0,1] using the weight function $w(x) = \ln x$.

**Hint:** $\int_0^1 x^n \ln(x)\, dx = \left[\frac{-1}{(n+1)^2}\right],\ n \geq 0.$ **[10 marks]**

## Question 2

(a) Assuming real $\lambda < 0$, find the interval of absolute stability for the methods

$$y_{k+2} = y_{k+1} + \frac{h}{2}[3f_{k+1} - f_k]$$

and hence conclude whether it is A-stable. **[10 marks]**

(b) Find values of $a$, $b$ and $c$ so as to ensure that the LMM

$$y_{k+3} = y_{k+2} + ahf_{k+2} + bhf_{k+1} + chf_k$$

is order 2. **[10 marks]**

## Question 3

(a) Convert the initial value problem

$$y'' - 2y + y = xe^x - x \qquad y(0) = y'(0) = 0,$$

whose true solution is $y(x) = \frac{1}{6}x^3e^x - xe^x + 2e^x - x - 2$, to a first-order system $\mathbf{u}' = \mathbf{f}(x, \mathbf{u})$. **[8 marks]**

(b) Apply the modified Euler method

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\left\{f(x_n, y_n) + f(x_{n+1}, \overline{y}_{n+1})\right\}, \\ \overline{y}_{n+1} &= y_n + hf(x_n, y_n) \end{aligned}$$

to the system obtained in (a) with $h = 0.1$ to obtain an approximation to $\mathbf{u}(0.1)$. **[12 marks]**

## Question 4

(a) Taking $h = \frac{1}{3}$, use the finite difference method to approximate the solution of the two-point b.v.p.

$$y'' = 4(y - x), \qquad 0 \le x \le 1, \qquad y(0) = 0, \qquad y(1) = 2.$$

**[10 marks]**

(b) Discuss consistency, zero stability and convergence of the linear multistep method

$$y_{n+2} = y_{n-2} + \frac{4h}{3}\{2f_{n+1} - f_n + 2f_{n-1}\}.$$

**[10 marks]**

## Question 5

Consider the standard initial value problem $y' = f(x, y)$, $y(0) = y_0$. We would like to construct a numerical method from the quadratic interpolant $P_2(x)$, of $f$ at the equally spaced nodes $x_{n-1}$, $x_n$ and $x_{n+1}$.

(a) Write down the Newton form of $P_2$ in forward difference form.

**[5 marks]**

(b) By integrating between $x_n$ and $x_{n+1}$, derive the implicit method

$$y_{n+1} = y_{n-1} + \frac{h}{3}\{f_{n-1} + 4f_n + f_{n+1}\}$$

**[7 marks]**

(c) Prove that this method is order 4, and find the leading term in the local truncation error.

**[8 marks]**

## Question 6

Consider the following partial differential equation:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0, \qquad 0 \le x \le 1, \qquad t > 0; \\
u(0,t) &= 0, \qquad u(1,t) = 0, \qquad t > 0, \\
u(x,0) &= \sin \pi x, \qquad 0 \le x \le 1, \\
\frac{\partial u}{\partial t}(x,0) &= 0, \qquad 0 \le x \le 1
\end{aligned}$$

By using the finite-difference method with $h = k = 1/3$, approximate the solution at $t = 2/3$ of the PDE, comparing your results with the actual solution of $u(x,t) = \cos \pi t \sin \pi x$ at $t = 2/3$. Assume the differential equation holds on the initial line, and use a second order Taylor series approximation to find an approximation to the solution at $t = 1/3$.

**[20 marks]**

## Question 7

(a) The following elliptic differential equation is for heat distribution on a rectangular plate $\{(x,y) : 0 \le x \le 1, \ 0 \le y \le 2\}$:

$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = 0, \quad u(x,2) = 0$$

$$u(0,y) = 0, \quad u(1,y) = 50$$

(i) Using $h = k = 1/2$, translate the problem into a corresponding difference problem based on the five-point formula, stating the boundary conditions in terms of the mesh points.

**[5 marks]**

(ii) Determine the system of equations to be used to solve the problem. (DO NOT SOLVE)

**[5 marks]**

7. (b) Consider the parabolic differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} &= 0, \qquad 0 \le x \le 1, \qquad t > 0 \\ u(0,t) &= 0, \qquad u(1,t) = 0, \qquad t > 0 \\ u(x,0) &= \cos 2\pi x, \qquad 0 \le x \le 1 \end{aligned}$$

(i) If an $O(k^2+h^2)$ numerical method is constructed using the central difference quotient to approximate $u_t$ and the usual difference quotient to approximate $u_{xx}$, show that the resulting difference problem is

$$u_{i,j-1} = u_{i,j-1} + 2\lambda(u_{i+1,j} - 2u_{ij} + u_{i-1j}).$$

defining $\lambda$ appropriately. **[4 marks]**

(ii) Give a sketch of the configuration of points involved in the computation of the solution at the internal grid point $(x_i, t_j)$. **[2 marks]**

(iii) Show that the method has the matrix form

$$\mathbf{u}^{(j+1)} = \mathbf{u}^{(j-1)} + A\mathbf{u}^{(j)} \quad \text{for each} \quad j = 0, 1, 2, \cdots$$

where $\mathbf{u}^{(j)} = (u_{1,j}, u_{2,j} \cdots, u_{m-1,j})^T$ and $A$ is a tridiagonal matrix.

**[4 marks]**

******** *END OF EXAMINATION* ********