UNIVERSITY OF SWAZILAND

Supplementary Examination 2005

Title of Paper : Numerical Analysis II

Program : BSc./B.Ed./B.A.S.S. IV

Course Number : M 411

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of SEVEN questions on THREE pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

Special Requirements: None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

Question 1

(a) Determine the straight line that best fits the following data in the least squares sense: (0,0), (2,2), (4,3) and (6,5).

[10 marks]

(b) Use the Gram-Schmidt procedure to calculate L_1 and L_2, where $\{L_o, L_1, L_2\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ and $L_o(x) = 1$.

[10 marks]

Question 2

(a) Show that the family of trigonometric functions $\{\phi_j(x)\}_{j=0}^{\infty}$, where $\phi_j(x) = \cos jx$, is orthogonal on $[0, 2\pi]$ with weight function $w(x) \equiv 1$. Also calculate $\|\phi_j\|$ for $j \geq 0$.

[10 marks]

(b) Hence, for $f(x) \in C[0, 2\pi]$, determine a_0 and a_1 such that $p_1 = a_0 + a_1 \cos x$ minimizes

$$\int_0^{2\pi} [f(x) - p_1(x)]^2 \, dx.$$

[10 marks]

Question 3

(a) Convert the initial value problem

$$y''' + 2y'' - y' - 2y = e^x, \qquad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 0,$$

to a first-order system $\mathbf{u}' = \mathbf{f}(x, \mathbf{u})$. **[8 marks]**

(b) Apply the modified Euler method

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\left\{f(x_n, y_n) + f(x_{n+1}, \overline{y}_{n+1})\right\}, \\ \overline{y}_{n+1} &= y_n + hf(x_n, y_n) \end{aligned}$$

to the system in (a) with $h = 0.1$ to obtain an approximation to $\mathbf{u}(0.1)$.

[12 marks]

Question 4

(a) Discuss convergence of the multi-step method

$$y_{k+4} = y_k + \frac{4h}{3}[2f_{k+3} - f_{k+2} + 2f_{k+1}]$$

[10 marks]

(b) For $f(x) \in C[-1,1]$, determine the polynomial $p_1 = a_0 + a_1 x$, of degree one that minimizes

$$\int_{-1}^{1} \frac{[f(x) - p_1(x)]^2}{\sqrt{1-x^2}}\,dx.$$

[10 marks]

Question 5

Consider the standard initial value problem $y' = f(x, y)$, $y(0) = y_0$. We would like to construct a numerical method from the quadratic interpolant $P_2(x)$, of f at the equally spaced nodes x_{n+1}, x_{n+2} and x_{n+3}.

(a) Write down the Newton form of P_2 in forward difference form.

[5 marks]

(b) By integrating between x_n and x_{n+4}, derive the method

$$y_{n+4} = y_n + \frac{4h}{3}\{2f_{n+3} - f_{n+2} + 2f_{n+1}\}$$

[7 marks]

(c) Prove that this method is order 4, and find the leading term in the local truncation error.

[8 marks]

Question 6

Consider the following hyperbolic partial differential equation:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad 0 \le x \le 1, \quad t > 0; \\
u(0,t) &= 0, \quad u(1,t) = 0, \quad t > 0, \\
u(x,0) &= 0, \quad 0 \le x \le 1, \\
\frac{\partial u}{\partial t}(x,0) &= 0, \quad 0 \le x \le 1
\end{aligned}$$

By using the finite-difference method with $h = k = 1/3$, approximate the solution at $t = 2/3$ of the PDE, comparing your results with the actual solution of $u(x,t) = \cos \pi x \sin \pi t$ at $t = 2/3$. Assume the differential equation holds on the initial line, and use a second order Taylor series approximation to find an approximation to the solution at $t = 1/3$. **[20 marks]**

Question 7

(a) Assuming real λ, determine the interval of absolute stability of the method

$$y_{k+2} = \frac{4}{3} y_{k+1} - \frac{1}{3} y_k + \frac{2h}{3} f_{k+2}$$

and hence conclude if it is A-stable. **[10 marks]**

(b) Consider the following Poisson equation over the square region $\{(x,y) : 0 < x < 1, \ 0 \le y \le 1\}$:

$$u_{xx} + u_{yy} = x$$

$$u(x,0) = u(x,1) = \frac{1}{6}x^3; \quad 0 \le x \le 1;$$

$$u(0,y) = 0, \quad u(1,y) = \frac{1}{6}; \quad 0 \le y \le 1$$

(i) Using $h = k = 1/3$, write down the corresponding difference problem based on the five-point formula, stating the boundary conditions in terms of the mesh points.

[5 marks]

(ii) Determine the system of equations to be used to solve the problem.

[5 marks]

******** *END OF EXAMINATION* ********