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University of Swaziland

Final Examination 2004/2005

B.Sc./B.Ed./B.A.S.S. IV

Title of Paper : **Partial Differential Equations**

Course Number : M 415

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of **seven questions.**
2. Answer **any five questions**.
3. Your work must be accompanied by appropriate explanations.
4. Use of **cellular phones** during the examination is not allowed.
5. Only non-programmable calculators may be used.

Special requirements: None

The examination paper must not be opened until permission has been granted by the Invigilator.

Q1.
Classify according to type and determine the characteristics of:

1. $u_{xx} - x^2 y u_{yy} = 0, (y > 0, x \neq 0)$
2. $e^{2x} u_{xx} + 2e^{x+y} u_{xy} + e^{2y} u_{yy} = 0$
3. $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0.$

20 [marks]

Q2.
Transform the equation $u_{xx} - 4u_{xy} + 4u_{yy} = e^y$ into canonical form and hence find its general solution. 20 [marks]

Q3.
The non-homogeneous wave equation is given by:

$$u_{tt} - c^2 u_{xx} = h(x,t); -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x), u_t(x,0) = g(x), -\infty < x < \infty.$$

Using the characteristic triangle, show that this equation has solution:

$$u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\int\int_{\Delta} h(x,t)d\Delta.$$

Note: Green's theorem: If M and N have continuous partial derivatives in an open region containing Δ, then $\int_C Mdx + Ndy = \int\int_{\Delta}(N_x - M_y)\,d\Delta$, where C is the boundary of Δ.

20 [marks]

Q4.
The vibrating string problem is described by the equation:

$$u_{tt} - c^2 u_{xx} = 0, 0 < x < l, t > 0,$$

$$u(x,0) = f(x), u_t(x,0) = g(x), 0 \leq x \leq l,$$
$$u(0,t) = u(l,t) = 0, t \geq 0.$$

Use the mothod of separation of variables to derive the solution:

$$u(x,t) = \sum_{n=1}^{\infty}\{a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\}\sin\frac{n\pi x}{l}.$$

Derive the formulae for a_n and b_n. Hint: Only the case $\sigma < 0$, where σ is the separation constant gives nontrivial solutions.

20 [marks]

Q5.
Consider the temperature distribution $u(x,t)$ within a homogeneous bar of length l described by the following equations:

$$u_t = ku_{xx}, 0 < x < l, t > 0,$$

$$u(x,0) = f(x), 0 \le x \le l,$$
$$u(0,t) = u(l,t) = 0, t \ge 0.$$

Use the method of separation of variables to derive the solution:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{l^2}kt}\sin\frac{n\pi x}{l}.$$

Derive the formula for a_n. Hint: Since the boundary conditions are homogeneous at $x = 0$ and $x = l$, the separation constant is taken to be $-\alpha^2$, where α is a positive constant.

[20 marks]

Q6
(a) Consider the temperature distribution $u(x,t)$ within a homogeneous bar of length l described by the following equations:

$$u_t = ku_{xx}, 0 < x < l, t > 0,$$

$$u(x,0) = f(x), 0 \le x \le l,$$
$$u(0,t) = T_1, u(l,t) = T_2, t \ge 0.$$

Assume the solution has the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is a steady state temperature distribution and hence solve the partial differential equation. Show that;

$$a_n = \frac{2}{l}\int_0^l \{f(x) - (T_2 - T_1)\frac{x}{l} - T_1\} \sin\frac{n\pi x}{l} dx.$$

Note: You may assume the resutl in Q5.
(b) Apply your solution to the problem:

$$u_t = u_{xx}, 0 < x < 30, t > 0$$

$$u(0,t) = 20, u(30,t) = 50, t \geq 0$$
$$u(x,0) = 60 - 2x, 0 < x < 30.$$

[20 marks]

Q7. Laplace's equation in a circle of radius a with a Dirichlet boundary condition is given by:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} = 0, 0 < r < a, -\pi < \phi \leq \pi$$

$$u(a,\phi) = f(\phi), -\pi < \phi \leq \pi,$$

where ϕ is the angular coordinate. Show that this equation has the solution:

$$u(r,\phi) = \frac{A_0}{2} + \sum_{n=0}^{\infty} r^n(A_n \cos n\phi + B_n \sin n\phi).$$

Derive formulae for A_0, A_n, B_n. Hint: The case $\sigma < 0$, where σ is the separation constant has no nontrivial solutions.

END OF PAPER