THE UNIVERSITY OF SWAZILAND 155

Department of Mathematics

Final Examination 2005

M423
ABSTRACT ALGEBRA II

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1. (a)[*7 marks*] Find the highest common factor of 10113 and 21671 and express it in the form $10113s + 21671t$

(b)[*6 marks*] Show that for any integers a, b

$$(3a + 2b, a + b) = (a, b)$$

(c) [*7 marks*] What is meant by saying that a polynomial $f(x) \in \mathbb{Q}[x]$ is *irreducible*? State Eisenstein's test for irreducibility, and use it to show that $33 + 15x + 21x^5 - 12x^7 + 9x^{10} - 14x^{11}$ is irreducible in $\mathbb{Q}[x]$.

Question 2. [*5 + 5 + 5 +5 marks*] Which of the following is a ring (with the usual operations)? For each *either* prove that it is a ring *or* explain why it is not.

(i) the set of 2×2 matrices of the form

$$\begin{pmatrix} 2a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are integers;

(ii) the set of rationals of the form $\frac{a}{b}$ where a, b are integers and $3 \nmid b$;

(iii) the set of all polynomials in $\mathbb{R}[x]$ of degree greater than two;

(iv) the set $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$.

(You may assume that $\mathbb{C}$ and $M_2(\mathbb{R})$, the set of all 2×2 matrices are both rings.)

Question 3. (a) [*7 marks*] What is (i) an *integral domain*; (ii) a *field* ? Show that a field is an integral domain. Give an example of an integral domain that is not a field.

(b) [*6 marks*] Which of the following rings is a field? Which is an integral domain? Give reasons for your answers.

(i) $\mathbb{R}[x]$ (ii) $\mathbb{Q}[i] = \{a + bi : a, b \in \mathbb{Q}\}$

(c) [*7 marks*] Explain what is meant by the ring $\mathbb{Z}_n$ where $n > 1$ is an integer. Show that $\mathbb{Z}_n$ is a field if and only if n is prime.

Question 4. (a)[*3 marks*] Define each of the following for u, r in a ring with unity.

(i) u is a unit; (ii) r is prime; (iii) r is irreducible.

(b) [*3 marks*] Show that in any integral domain a prime is irreducible.

(c) [*6 marks*] Let $a = 5 + 3i$ and $b = 1 + 2i$ in the ring $\mathbb{Z}[i]$ of Gaussian integers. Find q and r in $\mathbb{Z}[i]$ such that

$$a = qb + r$$

with $N(r) < N(b)$, where $N(m + ni) = m^2 + n^2$

(d) [*8 marks*] Outline the main steps in the proof that an irreducible element in $\mathbb{Z}[i]$ is prime.

CONT ...

Question 5. (a)[*9 marks*] Let d be a square free integer; define the *norm* $N(\alpha)$ of an element $\alpha \in \mathbb{Z}[\sqrt{d}]$. Show that

(i) $N(\alpha\beta) = N(\alpha)N(\beta)$; (ii) α is a unit if and only if $N(\alpha) = 1$.

(b) (i) [*3 marks*] Show that ± 1 are the only units in $\mathbb{Z}[\sqrt{-3}]$.

(ii) [*5 marks*] Show that $(1 \pm \sqrt{-3})$ and 2 are irreducible in $\mathbb{Z}[\sqrt{-3}]$.

(iii) [*3 marks*] By considering the product $(1+\sqrt{-3})(1-\sqrt{-3})$ show that $\mathbb{Z}[\sqrt{-3}]$ does not have the property of unique factorisation into irreducibles.

Question 6. (a) [*13 marks*] Let R and S be rings. What is meant by

(i) an *ideal* of R;

(ii) a *ring homomorphism* $\theta : R \to S$?

(iii) a *ring isomorphism* $\theta : R \to S$?

Define the *kernel* $\ker\theta$ of a ring homomorphism $\theta : R \to S$ and show that it is an ideal of R.

Show that a ring homomorphism $\theta : R \to S$ is an isomorphism if and only if it is surjective and $\ker\theta = \{0\}$.

(b) [*7 marks*] Which, if any, of the following is a ring homomorphism? Find the kernel for those that are homomorphisms.

(i) $\theta : \mathbb{R}[x] \to \mathbb{R}$ defined by $\theta(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + a_1 + a_2 + \cdots + a_n$;

(ii) $\theta : \mathrm{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by $\theta(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Question 7. (a) (i) [*3 marks*] Show that the polynomial $1 + x + x^3$ is irreducible in $\mathbb{Z}_2[x]$.

(b) Suppose that E is an extension field of $\mathbb{Z}_2$ and $\alpha \in E$ is a root of $1+x+x^3$.

(i) [*2 marks*] What is meant by the field $\mathbb{Z}_2(\alpha)$?

(ii) [*2 marks*] What is meant by the *minimum polynomial* of α over $\mathbb{Z}_2$? Explain why this is $1 + x + x^3$.

(iii) [*6 marks*] Show that every element of $\mathbb{Z}_2(\alpha)$ can be written uniquely as $a + b\alpha + c\alpha^2$ with $a, b, c \in \mathbb{Z}_2$.

(iv) [*7 marks*] Draw up the multiplication table for $\mathbb{Z}_2(\alpha)$ and identify the multiplicative inverse of each non-zero element.

(Any theorems you use about divisibility and HCFs in $\mathbb{Z}_2[x]$ should be stated clearly but not proved.)

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