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THE UNIVERSITY OF SWAZILAND

Department of Mathematics

Supplementary Examination 2005

M431
METRIC SPACES

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1. (a)[*10 marks*] What is meant by saying that (X, d) is a *metric space*?

Let d be the function defined on $\mathbb{R}^2$ by

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Prove carefully that $(\mathbb{R}^2, d)$ is a metric space.

(b) [*10 marks*] Describe the *uniform metric* and the L_2-*metric* on the set $C[a, b]$ of continuous functions defined on the interval $[a, b]$.

Let $x(t) = t$ and $y(t) = t^2$ for $-1 \leq t \leq 1$. Calculate the distance between x and y in $C[-1, 1]$

(i) in the uniform metric;

(ii) in the L_2-metric.

Question 2. (a) [*2 marks*] Let (X, d) be a metric space and (x_n) be a sequence in X. What is meant by saying that (x_n) is *convergent*?

(b)[*4+4 marks*] Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$.

(i) $x_n = \left(\frac{n^2+1}{2n^2+1}, \sin\left(\left(2n + \frac{1}{n}\right)\pi\right)\right)$ (ii) $x_n = (2^{-n}, \cos(n\pi))$

(c) [*10 marks*] Explain what is meant by *pointwise convergence* of a sequence (x_n) in $C[a, b]$. Show that if (x_n) converges in $C[a, b]$ with the uniform metric then (x_n) converges pointwise to the same limit.

Let x_n in $C[0, 1]$ be defined by

$$x_n(t) = \begin{cases} 1 - (n-1)t & \text{if } 0 \leq t \leq \dfrac{1}{n} \\ \dfrac{1-t}{n-1} & \text{if } \dfrac{1}{n} \leq t \leq 1 \end{cases}$$

Sketch the graph of $x_n(t)$ and show that (x_n) converges pointwise to the function

$$x(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } 0 < t \leq 1 \end{cases}$$

Deduce that (x_n) is not convergent in $C[0, 1]$ in the uniform metric.

CONT ...

Question 3. (a) [*5 marks*] Define what is meant by

(i) a *Cauchy sequence* in a metric space

(ii) a *complete metric space*.

Give an example of an incomplete metric space.

(b) [*6 marks*] Let (X, d) be a metric space with the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Show that any Cauchy sequence in X is eventually constant, and deduce that (X, d) is complete.

(c) [*9 marks*] Explain what is meant by a *contraction* of a metric space, and state without proof the *Contraction Mapping Theorem.*

Show that the mapping $f : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \frac{1}{6}(x^3 - x^2 + 3)$$

is a contraction, and deduce that there is a unique solution to the equation $x^3 - x^2 - 6x + 3 = 0$ in the interval $[-1, 1]$. (State any theorem you use to show that f is a contraction.)

Question 4. (a) [*6 marks*] Let (X, d) be a metric space and let $A \subseteq X$. What is meant by saying that A is *closed*? Show that if $(A_i)_{i \in I}$ is any collection of closed sets then the intersection $\bigcap_{i \in I} A_i$ is also closed.

(b)[*8 marks*] What is meant by a *closed ball* $B[a, r]$ in a metric space? Show that a closed ball is closed. By drawing a diagram or otherwise describe the closed ball $B[a, 2]$ in $\mathbb{R}^2$, where $a = (3, 3)$

(i) with the usual metric;

(ii) with the Chicago metric.

(c) [*6 marks*] Which of the following sets A is closed in the given metric space X.

(i) $X = \mathbb{R}^2$ (with the usual metric); $A = \{(a, b) : a + b = 0\}$

(ii) $X = C[0, 1]$ with the uniform metric; $A = \{x : x(\frac{1}{2}) < 2\}$

(iii) $X = \mathbb{R}$ with the usual metric; $A = \mathbb{Z}$.

CONT ...

Question 5. (a) [*10 marks*] Let $f : X \to Y$, where X and Y are metric spaces. Give the definition of f is *continuous* in terms of convergence of sequences. Show that if f is continuous then

(i) if U is an open subset of Y then $f^{-1}(U)$ is an open subset of X (say which definition of open you are using);

(ii) if $g : Y \to Z$ is also continuous, where Z is another metric space, then $g \circ f : X \to Z$ is continuous.

(b) [*4 marks*] Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous. Show that the function $h : \mathbb{R} \to \mathbb{R}^2$ defined by

$$h(x) = (f(x), g(x))$$

is continuous.

(c) [*6 marks*] Let f be the function $f : C[-1,1] \to \mathbb{R}$ defined for $x \in C[-1,1]$ by

$$f(x) = \max\{|x(t)| : -1 \le t \le 1\}$$

Show that f is not continuous with respect to the L_1 metric on $C[-1,1]$ (and the usual metric on $\mathbb{R}$) by considering the functions $x_n(t)$ given by

$$x_n(t) = \begin{cases} 1 - n|t| & \text{if } 0 \le |t| \le \frac{1}{n} \\ \\ 0 & \text{if } \frac{1}{n} \le |t| \le 1 \end{cases}$$

(**Hint** Sketch the functions $x_n(t)$ and consider their limit in the L_1 metric.)

Question 6. (a) [*4 marks*] Let X be a metric space and $A \subseteq X$. What is meant by saying that (i) A is *bounded* and (ii) A is *compact*?

(b) [*6 marks*] Show that a compact set is closed and bounded.

(c) [*4 marks*] Show that if X is a metric space with the discrete metric then any infinite set $A \subseteq X$ is closed and bounded but not compact.

(d) [*6 marks*] Which of the following sets is compact ?

(i) $\{(x,y) : -1 \le x \le y \le 1\}$ in $\mathbb{R}^2$ (with the usual metric)

(ii) $\mathbb{Q} \cap [-1,1]$ in $\mathbb{R}$ (with the usual metric).

Give reasons for your answers.

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Question 7. (a) [*3 marks*] Let d_1 and d_2 be metrics on a set X. What is meant by saying that the metrics d_1 and d_2 are *equivalent.*

(b) [*8 marks*] Suppose that there are positive constants k, K such that

$$kd_1(x,y) \leq d_2(x,y) \leq Kd_1(x,y)$$

for all $x, y \in X$. Show that if (x_n) is convergent in d_1 then it is convergent in d_2 and deduce that d_1 and d_2 are equivalent.

(c) [*4 marks*] Show that on $\mathbb{R}^2$ the usual (Euclidean) metric and the max metric are equivalent.

(d) [*5 marks*] Let d_1 and d_2 be equivalent metrics on X. Using the characterization of continuous functions in terms of open sets (which you should state clearly but need not prove), show that if $f : X \to X$ is continuous in the metric d_1 then it is continuous in the metric d_2.

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