# UNIVERSITY OF SWAZILAND

## Final Examination 2006

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**Title of Paper** : Linear Algebra

**Program** : BSc./B.Ed./B.A.S.S. II

**Course Number** : M 220

**Time Allowed** : Three (3) Hours

**Instructions** :

1. This paper consists of seven (7) questions on FOUR (4) pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

**Special Requirements** : None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

## Question 1

(a) Define the vector space. **[4 marks]**

(b) Show that the set $B = \{U_1, U_2, U_3, U_4\}$ where

$$U_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} \quad U_3 \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad U_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is a basis for $\mathbb{R}^4$. **[8 marks]**

(c) Verify the Cayley-Hamilton theorem for the following matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{pmatrix}.$$

**[8 marks]**

## Question 2

(a) Find a sequence of elementary matrices $E_1, E_2, \cdots, E_n$ such that

$A = E_n E_{n-1} \cdots E_2 E_1$ [i.e. $E_1^{-1} E_2^{-1} \cdots E_n^{-1} A = I$]:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

**[4 marks]**

(b) Find the inverses of the following matrices

(i) $\begin{pmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ **[4 marks]**

(ii) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ **[4 marks]**

2 (c) Use the results of Question 2(b) to solve the following systems of equations

(i)

$$\begin{aligned} 2x + 2y + z &= 1 \\ 3x + y + z &= 2 \\ x + y + z &= 2 \end{aligned}$$

**[4 marks]**

(ii)

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 - x_3 &= 1 \end{aligned}$$

**[4 marks]**

## Question 3

(a) Prove that if a homogeneous system has more unknowns than the number of equations, then it always has a non-trivial solution.

**[10 marks]**

(b) Set $B_1 = \{U_1, U_2, U_3\}$ and $B_2 = \{v_1, v_2, v_3\}$ be bases in $\mathbb{R}^3$, where

$$U_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad U_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

and

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find the transition matrix from $B_1$ to $B_2$.

**[10 marks]**

## Question 4

(a) By inspection, find the inverses of the following elementary matrices

(i) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. **[10 marks]**

(b) Prove that the set $B = \{x^2 + 1,\ x - 1,\ 2x + 2\}$ is a basis for the vector space $P_2(x)$ where
$P_2(x)$ – all polynomials of degree $\leq 2$ and the zero polynomial.
**[10 marks]**

## Question 5

(a) Solve the following systems
(i)

$$\begin{aligned} 2x + 2y + 3z &= 3 \\ 4x + 7y + 7z &= 1 \\ 4y - 2x + 5z &= -7 \end{aligned}$$

**[5 marks]**

(ii)

$$\begin{aligned} x_1 + 3x_2 - 2x_3 - 4x_4 &= 3 \\ 2x_1 + 6x_2 - 7x_3 - 10x_4 &= -2 \\ -x_1 - x_2 + 5x_3 + 9x_4 &= 14 \\ -3x_1 - 5x_2 + 15x_4 &= -16 \end{aligned}$$

**[10 marks]**

(b) Prove that if $A$ and $B$ are both invertible $n \times n$ matrices, then $AB$ is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$. **[5 marks]**

## Question 6

(a) Which of the following are linear transformations?

(i) $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 3y \\ 2x-y \end{pmatrix}$ **[5 marks]**

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ y \end{pmatrix}$ **[5 marks]**

(b) Find the standard matrices for the following linear transformations

(i) $T : \mathbb{R}^3 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y-z \end{pmatrix}$ **[5 marks]**

(ii) $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix}$ **[5 marks]**

## Question 7

(a) Find the eigenvalues and the corresponding eigenvectors for the following matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

**[10 marks]**

(b) For which $k$ does the following system have only the trivial solution?

$$\begin{aligned} kx + y - 3z &= 0 \\ (k-1)x + ky + z &= 0 \\ 3x + (k-1)y + kz &= 0 \end{aligned}$$

**[10 marks]**

******** *END OF EXAMINATION* ********