UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2006

BSC./B.Ed./B.A.S.S. II

TITLE OF PAPER: Linear Algebra

COURSE NUMBER: M220

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any FIVE questions.
3. Non-programmable calculators may be us

SPECIAL REQUIREMENTS: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1

(a) Find the adjoint of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

(b) Calculate def (A) and if A is invertible find $\det(a^{-1})$ where

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

c) Find the co-ordinate vector of $(1,5,9)^T$ with respect to $(1,0,0)^T, (1,1,0)^T, (1,1,1)^T$

[20 marks]

Question 2

(a) By inspection find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Define the notion of row equivalent matrices. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

(c) For which k does the following system have only the trivial solution?
(i)

$$\begin{aligned} kx_1 + x_2 - 3x_3 &= 0 \\ (k-1)x_1 + kx_2 + x_3 &= 0 \\ 3x_1 + (k-1)x_2 + kx_3 &= 0 \end{aligned}$$

[20 marks]

Question 3

(a) Solve the following system

$$\begin{aligned} x_1 - 2x_2 + x_3 - 4x_4 &= 1 \\ x_1 + 3x_2 + 7x_3 + 2x_4 &= 2 \\ x_1 - 12x_2 - 11x_3 - 16x_4 &= 5 \end{aligned}$$

(b) (i) Find the inverse of the matrix A and use A^{-1} to solve the system $A \cdot x = B$ where

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad x \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}$$

(ii) Find a finite sequence of elementary matrices $E_1, E_2, \cdots, E_n$ such that

$$E_n E_{n-1} \cdots E_2 E_1 A = I$$

[20 marks]

Question 4

(a)Which of the following transformations are linear?
(i) $T : P_2(x) \to P_1(x); T(ax^2 + bx + c) = 2ax + b$
(ii) $T : \mathbb{R}^3; T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+1 \\ 2y \\ z \end{pmatrix}$

(b) Let $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x+z \\ y-z \end{pmatrix}$ be a linear transformation.
(i) Find the matrix A of $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(ii) Find the matrix A' of T with respect to the std basis

(iii) Find a 3×3 transition matrix P from S to the std basis
(iv) Show that $A' = p^{-1}AP$.

[20 marks]

Question 5

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-2y \\ 2x+y \\ x+y \end{pmatrix}$

Find the matrix of T with respect to B_1 and B_2 where

$B_1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $B_2 \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \right\}$

(b) Which of the following sets of vectors span $\mathbb{R}^4$

(i) $(1,0,0,1)^T, (0,1,0,0)^T, (1,1,1,1)^T, (1,1,1,0)^T$

(ii) $(1,2,1,0)^T, (1,1,-1,0)^T, (0,0,0,1)^T$

[20 marks]

Question 6

(a) Let B_1 and B_2 be finite subsets of a vector space and let B_1 be a subset of B_2 Then show that

(i) if B_1 is linearly dependent, so is B_2

(ii) if B_2 is linearly independent, so is B_1

(b) Find the matrix of T with respect to the given basis

(i) $T : \mathbb{R}^3 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x+2y+3z \end{pmatrix}$

$B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $B_2 \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$,

(ii) $T : P_2(x) \to P_1(x)$; $T(T(x) = P'(x)$

$B_1 = \{x^2, x, 1\}$ and $B_2 = \{x, 1\}$

$P - n(x)-$ all polynomials of degree $\leq n$ and the zero polynomial

[20 marks]

Question 7

(a) Explain whether the following statement is true: if a triangular matrix (upper of lower) is invertible then its diagonal entries are all nonzero

(b) Determine whether the linear transformation T is one to one

(i)$T : \mathbb{R}^3 \to \mathbb{R}^3$; $\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$

(ii)$T : \mathbb{R}^3 \to \mathbb{R}^2$; $\quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

(c) Prove that if a homogeneous system has more unknowns than the number of equations then it always has a non-trivial solution.

[20 marks]

******** *END OF EXAMINATION* ********