UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2006

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Prove that in any set of $n+1$ pairwise distinct integers, there must be two whose difference is divisible by n. [7]

(b) Prove, by the contrapositive method, that if no angle of a quadrilateral $RSTU$ is obtuse, then the quadrilateral $RSTU$ is a rectangle. [6]

(c) (i) Show that if r is a nonzero rational number, then $r\sqrt{7}$ is an irrational number. [4]

(ii) Using the result in (a), or otherwise, show that $\sqrt{28}$ is irrational. [3]

QUESTION 2

(a) Let p_1 and p_2 be distinct prime numbers. Prove that the real numbers $\sqrt{p_1}+\sqrt{p_2}$ and $\sqrt{p_1}-\sqrt{p_2}$ are irrational. [10]

(b) Prove that the square root of a natural number is rational if and only if the natural number is a perfect square. [10]

QUESTION 3

(a) (i) Define a square-free natural number. [2]

(ii) Let b and m be two natural numbers such that b is square-free and m^2 is divisible by b. Prove that m is also divisible by b. [8]

(b) Prove that the square root of any prime number is irrational. [10]

QUESTION 4

(a) Suppose you want to show that $A \Rightarrow B$ is **false**. How should you do this? What should you try to show about the truth of A and B? [2]

(b) Apply your answer of part (a) to show that the statement "If x is a real number that satisfies $-3x^2 + 2x + 8 = 0$, then $x > 0$" is false. [3]

(c) Write the negation of the statement: "The real-valued function f of one variable is *continuous at the point* x if and only if for every real number $\varepsilon > 0$, there is a real number $\delta > 0$ such that, for all real numbers y with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$." [5]

(d) Prove that if a, b and c are integers for which either a divides b or a divides c, then a divides the product bc. [4]

(e) Work out 1, $1+8$, $1+8+27$, $1+8+27+64$. Guess a formula for $\sum_{r=1}^{n} r^3$ and prove it. [6]

QUESTION 5

(a) Describe a modified induction procedure that could be used to prove statements of the form:

(i) For all integers $n \leq k$, $P(n)$ is true, where $P(n)$ is a statement containing the integer n. [3]

(ii) For all integers n, $P(n)$, where $P(n)$ is as in (a). [4]

(iii) For every positive odd integer, something happens. [3]

(b) For all non-negative integers n define the number u_n inductively as

$$\begin{aligned} u_0 &= 0, \\ u_{k+1} &= 3u_k + 3^k \qquad \text{for } k \geq 0. \end{aligned}$$

Prove that $u_n = n3^{n-1}$ for all non-negative integers n. [4]

(c) If $f(n) = 3^{2n} + 7$, where n is a natural number, show that $f(n+1) - f(n)$ is divisible by 8. Hence prove by induction that $3^{2n} + 7$ is divisible by 8. [6]

QUESTION 6

(a) Prove that a real number is rational if and only if its decimal representation is repeating. [10]

(b) Suppose that $a_0.a_1a_2a_3\ldots$ and $b_0.b_1b_2b_3\ldots$ are two different decimal representations of the same real number. Prove that one of these expressions ends in $9999\ldots$ and the other in $0000\ldots$. [10]

QUESTION 7

(a) State and prove the Fundamental Theorem of Arithmetic. [12]

(b) Prove that there are infinitely many primes of the form $4k + 3$, where k is a nonnegative integer. [8]

END OF EXAMINATION