UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2006

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) State the difference between deductive reasoning and inductive reasoning. Which of the two is a valid form of argument? Explain. [4]

(b) Prove that if n is an integer and n^2 is divisible by 2, then so is n. [6]

(c) Using the result in part (b), or otherwise, prove that if r is a real number such that $r^2 = 2$, then r is irrational. [10]

QUESTION 2

(a) Write the negation of the following statement: "The real-valued function f of one variable is *bounded above* if and only if there is a real number y such that for every real number x, $f(x) \leq y$." [6]

(b) Show that if $x > 2$ is a real number, then there is a unique real number $y < 0$ such that $x = \dfrac{4y}{2 + 2y}$. [6]

(c) Express $1.813813813\cdots$ as a fraction $\dfrac{m}{n}$, where m and n are integers. [5]

(d) Is the real number $0.101001000100001000001\cdots$ rational or irrational? Support your answer. [3]

QUESTION 3

(a) (i) Define a square-free natural number. [2]

(ii) Let b and m be two natural numbers such that b is square-free and m^2 is divisible by b. Prove that m is also divisible by b. [8]

(b) Suppose n rings, with different outside diameters, are slipped onto an upright peg, the largest ring at the bottom, the second largest on top of it, and so on, so that the smallest ring is at the top, to form a pyramid. Two other upright pegs are placed sufficiently far apart. We wish to transfer all the rings, one at a time, to the second peg to form an identical pyramid. During the transfers, we are not permitted to place a larger ring on top of a smaller one (which necessitates the third peg). What is the smallest number of moves necessary to complete the transfer? [10]

QUESTION 4

(a) The *Fibonacci sequence* is a sequence of integers $u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$, such that $u_1 = 1$, $u_2 = 1$ and

$$u_{n+1} = u_n + u_{n-1}$$

for all $n \geq 1$. The beginning of this sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. Prove by strong induction that for all positive integers n,

$$u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),$$

$$\text{where} \quad \alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

. [12]

(b) Suppose that Canada Post prints only 3 cent and 5 cent stamps. Prove that it is possible to make up any postage of n cents using only 3 cent and 5 cent stamps for $n \geq 8$. [8]

QUESTION 5

(a) Let $x = 0.a_1a_2a_3\ldots$, where for $n = 1, 2, 3, \ldots$, the value of a_n is the number 0, or 1, or 2, or 3 which is the remainder on dividing n by 4. Is x rational? If so, express x as a fraction $\frac{m}{n}$ where m and n are integers with $n \neq 0$. [8]

(b) Prove that between any two different irrational numbers there is a rational number and an irrational number. [12]

QUESTION 6

(a) Let $a, d \in \mathbb{N}$, with $d \geq 2$. Show that if d divides a, then d does not divide $a + 1$. [4]

(b) Prove that there are infinitely many primes. [8]

(c) Show that the square root of a natural number is rational if and only if the natural number is a perfect square. [8]

QUESTION 7

(a) State and prove the Fundamental Theorem of Arithmetic. [12]

(b) Prove that there are infinitely many primes of the form $3k + 2$, where k is an integer. [8]

END OF EXAMINATION