UNIVERSITY OF SWAZILAND

Final Examination 2006

Title of Paper	:	Numerical Analysis I
Program	:	BSc./B.Ed/B.A.S.S. III
Course Number	:	M 311
Time Allowed	:	Three (3) Hours
Instructions	:	

1. This paper consists of seven (7) questions on FOUR (4) pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

Special Requirements : None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

Question 1

(a) Let $x = (11111....11)_2$ be a binary number with n 1s. Convert x into its decimal equivalent.

[5 marks]

(b) Demonstrate how you would reformulate the following computation so as to avoid loss-of-significance error:

$$f(x) = \frac{e^{x} - e^{-x}}{2x}, \quad x \simeq 0$$

[5 marks]

(c) The iteration $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ will converge for sufficiently close x_0 to $s = 1$ for some values of c. Find the values of c for which this is true. For what value of c will the convergence be quadratic?

[10 marks]

Question 2

(a) Consider the iterative scheme

$$x_{n+1} = \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a}, \qquad a > 0.$$

(i) Show that $s = \sqrt{a}$ is the positive fixed point of this scheme.

[3 marks]

(ii) Assuming convergence to s, find the order of this method, together with its asymptotic error constant.

[5 marks]

(b) Factor the matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ into its LU decomposition, and hence solve the linear system

$$\begin{aligned} 2x_1 - x_2 \phantom{{}+2x_3} &= 3 \\ -x_1 + 2x_2 - x_3 &= -5 \\ - x_2 + 2x_3 &= 5 \end{aligned}$$

[12 marks]

Question 3

(a) Let s be a root of multiplicity $p \geq 2$ of f, where f is continuous, together with its first $p+1$ derivatives. Prove that the fixed point method

$$x_{n+1} = x_n - p\,\frac{f(x_n)}{f'(x_n)},$$

converges quadratically, and find its asymptotic error constant.

[10 marks]

(b) The function $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2 = (x-1)^3(x-2)$ has roots $\bar{x}_1 = 1$ and $\bar{x}_2 = 2$. Using $x_0 = 2.1$ and $x_0 = 0.9$, perform one step of the Newton-Raphson method for each starting value. Compute $|s - x_1|$ in each case.

Apply the secant method once with $x_0 = 0.9$ and $x_1 = 1.1$ to obtain x_2. Compute $|\ \bar{x}_1 - x_2\ |$, and briefly explain your observations.

[10 marks]

Question 4

Suppose an approximation to $\int_0^{2h} f(x)\,dx$ is sought, and $f(x)$ in $[0,\ 2h]$ is approximated by the linear function through the TWO points $(0, f(0))$, and $(h,\ f(h))$.

(i) Write down the Lagrange representation of the polynomial that interpolates f at the two points $(0, f(0))$, and $(h,\ f(h))$.

[4 marks]

(ii) By integrating the polynomial in (i) above between 0 and $2h$, prove that the desired quadrature formula is simply

$$I \approx \tilde{I} = 2h\,f(h).$$

[8 marks]

(iii) Show, using Taylor series expansions about a suitable point, and assuming $f \in C^2[0, 2h]$, that the quadrature error is given by

$$I - \tilde{I} = \frac{h^3}{3} f''(c), \qquad c \in [0,\ 2h].$$

[8 marks]

Question 5

(a) Use the definition of the derivative at x_0 to show that if h is sufficiently small, then

$$f'(x_0) \approx \frac{1}{h}\Delta f(x_0).$$

Extend this argument to show that

$$f''(x_0) \approx \frac{1}{h^2}\Delta^2 f(x_0).$$

[10 marks]

(b) Given the data

x	$f(x)$
-2	-1
-1	3
0	1
1	-1
2	3

Construct a forward-difference table, and hence deduce the polynomial of degree ≤ 4 that interpolates f at these points.

[10 marks]

Question 6

(a) Evaluate the integral $\int_0^1 xe^{-x}\,dx$ analytically correct to four decimal places. Use the trapezoidal rule with $h = 0.2$ and Simpson's rule with $h = 0.25$ to compute the same integral. Compare the errors.

[10 marks]

(b) Find constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_0^1 f(x)\,dx = c_0 f(0) + c_1 f(x_1)$$

is exact for polynomials of as high a degree as possible.

[10 marks]

Question 7

(a) Use the two-point Gaussian Quadrature rule,

$$\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right),$$

to approximate the integral

$$\int_{0}^{1} x^2 e^{-x}\,dx.$$

and compare your result against the exact value of the integral.

[10 marks]

(b) The positive root of $f(x)=\alpha-\beta x^2-x$ with $\alpha,\ \beta>0$ is sought and the simple iteration $x_{n+1}=\alpha-\beta x_n^2$ is used. Show that convergence will occur for sufficiently close starting value, provided

$$\alpha\beta<\frac{3}{4}.$$

[10 marks]

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