UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2006

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Part of a railway line (superimposed on a rectangular coordinate system) follows the line $y = -x$ for $x \leq 0$, then turns to reach the point (4,0) following a cubic curve. Find the equation of this curve if the track is *continuous*, *smooth*, and has *continuous curvature*. [8]

(b) (i) Find the scale h_1, h_2, and h_3 in cylindrical and in spherical coordinates. Hence find the volume element dV (in cylindrical and in spherical coordinates. [9]

(ii) Show that the spherical coordinate system is orthogonal. [3]

QUESTION 2

(a) Let $\mathbf{u}(x, y, z) = x\hat{\mathbf{i}} - y\hat{\mathbf{j}}$ and $\mathbf{v}(x, y, z) = \dfrac{\mathbf{u}}{(x^2 + y^2)^{\frac{1}{2}}}$ be vectors in space.

(i) Compute the divergence and the curl of $\mathbf{u}$ and $\mathbf{v}$. [6]

(ii) Find the flow lines of $\mathbf{u}$ and $\mathbf{v}$. [8]

(b) Determine the directional derivative of $\phi(x, y) = 100 - x^2 - y^2$ at the point (3,6) in the direction of the unit vector $\hat{\mathbf{u}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. [6]

QUESTION 3

(a) Find the tangent plane and the normal line to the surface $x^2y + xyz - z^2 = 2$ at the point $P_0(1, 1, 3)$. [10]

(b) Show that $\mathbf{n}(t) = -g'(t)\hat{\mathbf{i}} + f'(t)\hat{\mathbf{j}}$ and $-\mathbf{n}(t) = g'(t)\hat{\mathbf{i}} - f'(t)\hat{\mathbf{j}}$ are both normals to the curve $\mathbf{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ at the point $(f(t), g(t))$. Hence find $\hat{\mathbf{N}}$ for the curve $\mathbf{r}(t) = \sqrt{4 - t^2}\,\hat{\mathbf{i}} + t\hat{\mathbf{j}}, \quad -2 \leq t \leq 2$.. [10]

QUESTION 4

(a) By any method, find the integral of $H(x, y, z) = yz$ over the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the cone $z = \sqrt{x^2 + y^2}$. [7]

(b) Find the work done in moving a particle in the counterclockwise direction once around the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ if the force field is given by $\mathbf{F} = (3x - 4y)\hat{\mathbf{i}} + (4x + 2y)\hat{\mathbf{j}} - 4y^2\hat{\mathbf{k}}$. [6]

(c) Show that $y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z$ is exact and evaluate the integral

$$\int_{(2,2,2)}^{(3,4,0)} y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z.$$

[7]

QUESTION 5

(a) Find out which of the fields given below are conservative. For conservative fields, find a potential function.

(i) $\mathbf{F} = (yz^2)\hat{\mathbf{i}} + (xz^2)\hat{\mathbf{j}} + (x^2yz)\hat{\mathbf{k}}$.

(ii) $\mathbf{F} = (e^x \sin y)\hat{\mathbf{i}} + (e^x \cos y + \sin z)\hat{\mathbf{j}} + (y \cos z)\hat{\mathbf{k}}$. [12]

(b) Integrate $f(x, y, z) = 2x - 6y^2 + 2z$ over the line segment C joining the points (2,2,2) and (3,3,3). [8]

QUESTION 6

(a) By any method, find the outward flux of the field $\mathbf{F} = (6x^2 + 2xy)\hat{\mathbf{i}} + (2y + x^2z)\hat{\mathbf{j}} + (4x^2y^3)\hat{\mathbf{k}}$ across the boundary of the region cut from the first octant by the cylinder $x^2 + y^2 = 9$ and the plane $z = 9$. [10]

(b) By any method, find the circulation of the field $\mathbf{F} = (x^2 + y^2)\hat{\mathbf{i}} + (x + y)\hat{\mathbf{j}}$ around the triangle with vertices (1,0), (0,1), (-3,0) traversed in the counterclockwise direction. [10]

QUESTION 7

(a) Verify the divergence theorem for $\mathbf{F} = (2x - z)\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} - xz^2\hat{\mathbf{ki}}$ taken over the region bounded by $x = 2$, $x = 5$, $y = 2$, $y = 5$, $z = 2$, $z = 5$. [10]

(b) Verify Green's theorem in the plane for

$$\oint_C [2x\mathrm{d}x - (3y - x)\mathrm{d}y],$$

where C is the closed curve (described in the positive direction) of the region bounded by the curves $y = x^2$ and $y^2 = x$. [10]

END OF EXAMINATION