THE UNIVERSITY OF SWAZILAND

Department of Mathematics

Final Examination 2006

M313

COMPLEX ANALYSIS

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Throughout this paper the symbols $\mathbb{R}, \mathbb{C}$ stand for the real numbers and the complex numbers respectively.

Question 1. (a) [*7 marks*] Find all solutions to the equation $z^4 = -4$, expressing them in both rectangular and polar forms. Indicate their position in the complex plane.

(b) [*6 marks*] State *de Moivre's Theorem* and use it to prove the identity

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

(c) [*7 marks*] Describe the set of values of z for which $|z - \frac{3}{2}i| = \frac{3}{2}$ and show that it is the same as the set of values of z for which

$$\frac{|z-i|}{|z+3i|} = \frac{1}{3}$$

Question 2. (a) [*4 marks*] Let $f(z)$ be a complex function. What is meant by saying that f is *differentiable* at a point $z_0 \in \mathbb{C}$? What is meant by saying that f is *analytic* in an open set $S \subseteq \mathbb{C}$?

(b) [*4 marks*] Show that the function $f(z) = |z|^2$ is differentiable at $z_0 = 0$.

(c) [*4 marks*] State the *Cauchy-Riemann equations* for a complex function $f(z) = u(x,y) + iv(x,y)$ that is (complex) differentiable at a point $z_0 \in \mathbb{C}$.

(d) [*4 marks*] Find $u(x,y)$ and $v(x,y)$ for the function $f(z) = |z|^2$ and deduce that this function is **not** differentiable at any point $z_0 \neq 0$.

(e) [*4 marks*] Let $u(x,y) = x(1+2y)$. Find a function $v(x,y)$ such that the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic.

Question 3. (a) [*6 marks*] Give the definition of the complex exponential function $\exp(z) = e^z$ (where $z = x + iy$) and show that it obeys the property

$$e^{z_1}e^{z_2} = e^{z_1+z_2}$$

(You may assume the basic properties of the real exponential function $\exp x = e^x$ for $x \in \mathbb{R}$; any other results you use should be stated clearly)

Show that $e^{\bar{z}} = \overline{e^z}$ for all z.

(b) [*6 marks*] (i) Explain the meaning of the complex logarithmic function $\log z$. Show that $\exp(\log z) = z$ for every value of $\log z$.

(ii) What is meant by the *principal value* $\mathrm{Log} z$ for $z \in \mathbb{C}$? Show by an example that it is not always true that $\mathrm{Log}(e^z) = z$.

(c) [*8 marks*] Show that

(i) $\log i = (2n + \frac{1}{2})\pi i \quad (n = 0, \pm 1, \pm 2, ...)$

(ii) $\log(-1 + \sqrt{3}i) = \ln 2 + 2(n + \frac{1}{3})\pi i \quad (n = 0, \pm 1, \pm 2, ...)$

CONT ...

Question 4. (a) [*4 marks*] State without proof the *Cauchy-Goursat Theorem* and *Cauchy's integral formula* for an analytic function and its derivatives. [Ensure that you state clearly the conditions needed to make your statements true.]

(b) [*16 marks*] Use the above to evaluate the following:

(i) $\int_C \frac{z^2 \cos \pi z}{z+3} dz$ where C is the circle $|z| = 2$ traversed anticlockwise.

(ii) $\int_C \frac{z^2 \cos \pi z}{z+3} dz$ where C is the circle $|z| = 4$ traversed anticlockwise.

(iii) $\int_C \frac{z^2 \cos \pi z}{(z+3)^2} dz$ where C is the circle $|z| = 4$ traversed anticlockwise.

(iv) $\int_C \frac{z^2}{z^2+3} dz$ where C is the circle $|z - i| = 2$ traversed anticlockwise.

Question 5. (a) [*10 marks*] State without proof *Liouville's Theorem*. Use it to prove the *Fundamental Theorem of Algebra*: any non-constant polynomial $p(z)$ (i.e. $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$ with $n \geq 1$ and $a_n \neq 0$) has at least one zero (i.e. there is at least one point $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$).

(b) [*10 marks*] (i) State *Taylor's Theorem* for a complex function $f(z)$. Show that the function $f(z) = \frac{1}{z}$ is analytic in the disc $|z - 1| < 1$ and find its Taylor series about the point $z_0 = 1$.

(ii) For z with $|z - 1| > 1$ let $w = \frac{z}{z-1}$. Show that $w - 1 = \frac{1}{z-1}$ so that $|w - 1| < 1$. Show further that $\frac{1}{z} = \frac{1}{z-1} \cdot \frac{1}{w}$ and use the result of (i) to show that the function $f(z) = \frac{1}{z}$ has Laurent series

$$\frac{1}{z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(z-1)^n}$$

valid in the region $D = \{z : |z - 1| > 1\}$.

Question 6. (a) [*10 marks*] Let $f(z)$ be a complex function and $z_0 \in \mathbb{C}$. What is meant by saying that (i) z_0 is a *singular point* (or *singularity*) of f (ii) z_0 is an *isolated singularity* of f (iii) z_0 is a *pole of order m* (where $m \geq 1$).

(b) [*10 marks*] Describe all the poles, and find the corresponding residues, of the following functions:

(i) $f(z) = \frac{e^z}{z^2 + 16}$ (ii) $f(z) = \frac{z^3 + z}{(z + i)^3}$

(Any theorems you use should be stated clearly.)

Question 7. (a) [*20 marks*] Use the residue theorem and a suitable contour integral to show that

$$\int_0^{\infty} \frac{x^2 + 1}{x^4 + 1} = \frac{\pi}{\sqrt{2}}$$

(You may use without proof the fact that if $z_0^4 + 1 = 0$ then

$$z^4 + 1 = (z - z_0)(z^3 + z_0 z^2 + z_0^2 z + z_0^3)$$

for all z).

(END)