UNIVERSITY OF SWAZILAND

Final Examination 2006

Title of Paper : Abstract Algebra I

Program : BSc./B.Ed./B.A.S.S. III

Course Number : M 323

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of seven (7) questions on THREE (3) pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

Special Requirements : None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

Question 1

(a) Find all the subgroups of $\mathbb{Z}_{18}$ and draw the lattice diagram. **[10 marks]**

(b) Let G and H be groups, $\varphi : G \to H$ be an isomorphism of G and H and let e be the identity element of G. Prove that $(e)\varphi$ is the identity in H and that $[(a)\varphi]^{-1} = (a^{-1})\varphi$ for all $a \in G$. **[10 marks]**

Question 2

(a) Prove that a non-abelian group of order $2p$, p prime contains at least one element of order p. **[6 marks]**

(b) Consider the following permutations in S_6

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

Compute

(i) $\rho\sigma$ (ii) ρ^2 (iii) ρ^{-1} (iv) ρ^{-2} (v) $\sigma\rho^2$

[10 marks]

(c) Write the permutations in (b) as products of disjoints cycles in S_6.

[4 marks]

Question 3

(a) (i) State Cayle's theorem. **[4 marks]**

(ii) Let $(\mathbb{R}^+, \cdot)$ be the multiplicative group of all positive real numbers and $(\mathbb{R}, +)$ be the additive group of real numbers. Show that $(\mathbb{R}^+, \cdot)$ is isomorphic to $(\mathbb{R}, +)$.

[6 marks]

(b) (i) Find the number of generators in each of the following cyclic groups $\mathbb{Z}_{30}$ and $\mathbb{Z}_{42}$. **[5 marks]**

(ii) Determine the right cosets of $H = \langle 4 \rangle$ in $\mathbb{Z}_8$. **[5 marks]**

Question 4

(a) Show that $\mathbb{Z}_p$ has no proper subgroup if p is prime.

[6 marks]

(b) Show that if $(a,\ m) = 1$ and $(b,\ m) = 1$ then $(ab,\ m) = 1, \quad a, b, m \in \mathbb{Z}$.

[6 marks]

(c) Prove that every group of prime order is cyclic.

[8 marks]

Question 5

(a) (i) Define the notion of a "normal subgroup" of a group.

[4 marks]

(ii) Verify that $H = \{(1),\ (123),\ (132)\}$ is a normal subgroup of S_3.

[6 marks]

(b) Prove that every subgroup of a cyclic group is cyclic.

[10 marks]

Question 6

(a) Show that $(\mathbb{Z}_7^0,\ \cdot)$ is cyclic and give all generators of the group.

[5 marks]

(b) Prove that, if the order of a group G is p^2, where p is prime, then every proper subgroup of G is cyclic.

[5 marks]

(c) (i) Express $d = (2190,\ 465)$ as an integral linear combination of 219 and 465.

[5 marks]

(ii) Solve the following

$$3x \equiv 5 (\text{mod } 11)$$

[5 marks]

Question 7

(a) For each binary operation $*$ defined on a set G, say whether or not $*$ gives a group structure on the set.

(i) Define $*$ on $G = \mathbb{Q}^+$ by

$$a * b = \frac{ab}{2} \quad \forall \quad a,\ b \in G = Q^+.$$

[5 marks]

(ii) Define $*$ on $G = \mathbb{R}$ by

$$a * b = ab + a + b \quad \forall \quad a, b \in G = \mathbb{R}.$$

[5 marks]

(b) Show that $\mathbb{Z}_6$ and S_3 are NOT isomorphic and that $\mathbb{Z}$ and $n\mathbb{Z}$ are isomorphic.
[10 marks]

******** *END OF EXAMINATION* ********