UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2006

B.Sc. III/B.Ed./B.A.S.S. III

TITLE OF PAPER: ABSTRACT ALGEBRA

COURSE NUMBER: M323

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any FIVE questions.
3. Non-programmable calculators may be us

SPECIAL REQUIREMENTS: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1

(a) Find (a, b) and $[a, b]$ by first decomposing (writing) as a product of primes [10]

$$a = 144 \qquad b = 1250$$

(b) Solve the system [10]

$$3x \equiv 2(mod 5)$$

$$2x \equiv 1(mod 3)$$

Question 2

(a) Prove that if G is a group and that $\forall a \in G \quad a^2 = e$ then G is abelian [10]
(b) For a group G define the following relation for $a, b, \in G$
"$aRb \iff$ there exists $x \in G$ such that $b = x^{-1}ax$"
Show that the above relation is an equivalence relation. [10]

Question 3

(a) (i) Find all the conjugate elements of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ in S_3 [5]

(ii) Determine the order of
(1346)(287) in S_8 [5]

(b) Prove that every group of prime order is cyclic [10]

Question 4

(a) Find all subgroups of $\mathbb{Z}_{12}$ and draw the lattice diagram [10]

(b) For each binary operation $*$ defined on a set G, say whether or not $*$ gives a group structure on the set

$$G = \mathbb{Q} \qquad \text{and} \qquad a * b = a + b - 2006 \forall a, b \in G$$

[10]

Question 5

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 7 & 8 & 5 & 6 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 7 & 6 & 2 & 8 & 4 & 3 \end{pmatrix}$

(a) Express α and β as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even permutation or an odd one. [6]

(b) Compute $\alpha^{-1}, \beta^{-1} \circ \alpha$ and $(\alpha \circ \beta)^{-1}$ [9]

(c) Find the order of β. [5]

Question 6

(a) Let $H = < 8 >$ be the subgroup of $\mathbb{Z}_{20}$ generated by the element 8. Find all cosets of H in $\mathbb{Z}_{20}$ [10]

(b) Prove that every subgroup of a cyclic group is cyclic [10]

Question 7

(a) Let $\varphi : G \to H$ be an isomorphism of groups.

(i) Prove that if e_G and e_H are the identity elements of G and H respectively, then [6]

$$\varphi(e_G) = e_H$$

(ii) $[\varphi(a)]^{-1} = \varphi(a^{-1}) \quad \forall a \in G.$ [6]

(b) Given an example of a group satisfying the given conditions or, if there is no such group, say so. (Do not prove anything)

(i) A cyclic group of order 4

(ii) A non-abelian group of order 5

(iii) An infinite cyclic group

(iv) A non-abelian cyclic group [8]

******** *END OF EXAMINATION* ********