THE UNIVERSITY OF SWAZILAND

Department of Mathematics

Final Examination 2006

M331
REAL ANALYSIS

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Throughout this paper the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ stand for the natural numbers, the integers, the rational numbers and the real numbers respectively.

Question 1. Let A be a subset of the real numbers.

(a) [*10 marks*] What is meant by saying that A is *bounded above*? What is meant by the *supremum of* A (if it exists)?

State the *Completeness Property* of $\mathbb{R}$ in terms of the existence of suprema. Use it to derive the *Archimedean Property* of $\mathbb{R}$:

for any $x \in \mathbb{R}$ there is a natural number $n \in \mathbb{N}$ with $n > x$.

(b) [*6 marks*] Which of the following sets is bounded above? For those that are find the supremum. Give reasons for your answers.

(i) $\{\frac{1}{n^2+1} - n^2 : n \in \mathbb{Z}\}$

(ii) $\left\{q^2 - \frac{1}{q^2+1} : q \in \mathbb{Q}\right\}$

(c) [*4 marks*] Which of the following statements is always true? Give a proof for those that are true and a counterexample for those that are false.

(i) if A is not bounded above then $\mathbb{R} \setminus A$ is bounded above;

(ii) if A is bounded above then $\mathbb{R} \setminus A$ is not bounded above.

Question 2. (a) [*7 marks*] Let (a_n) be a sequence of real numbers. Give a precise definition of the statement that (a_n) is *convergent.*

Show **directly from the definition** that the sequence

$$\frac{n+1}{(\sqrt{n}+1)^2}$$

converges to 1.

(b) [*4 marks*] Prove carefully that a convergent sequence is bounded.

(c) [*9 marks*] Which of the following sequences (a_n) is convergent? For those that are, find the limit. State clearly any facts about limits that you use.

(i) $a_n = \dfrac{3n^3 - n^2 + 2}{n - 17n^2}$

(ii) $a_n = \dfrac{n - n^2 + 2n^3}{5n^4 - 17n} . \sin(n^2)$

(iii) $a_n = \sqrt{n^2+1} - \sqrt{n^2-1}$

CONT ...

Question 3. (a) [*4 marks*] Let (a_n) be a sequence. (i) Define what is meant by the *partial sums* of the series $\sum a_n$ (ii) What is meant by saying that $\sum a_n$ is convergent and that $\sum_{n=1}^{\infty} a_n = s$.

(b) [*3 marks*] Show that if $\sum |a_n|$ is convergent then $\sum a_n$ is also convergent.

(c) [*6 marks*] Show that the series $\sum \frac{1}{n}$ is divergent.

(d) [*4 marks*] Show that the series $\sum (-1)^n \frac{1}{n}$ is convergent, and deduce that the converse of (b) is false.

(e) [*3 marks*] Show that the series $\sum \frac{2^n}{n^n}$ is convergent

(For parts (d) and (e) state any general theorems that you use.)

Question 4. (a) [*6 marks*] Find $\lim_{x\to c} f(x)$ for each of the following functions and the given value of c.

(i) $$f(x) = \begin{cases} \dfrac{x^2-9}{x^3-27} & \text{if } x \neq 3 \\ 18 & \text{if } x = 3 \end{cases} \quad ; \quad c = 3$$

(**Hint:** $a^3 - b^3 = (a-b)(a^2+ab+b^3)$)

(ii) $$f(x) = \begin{cases} (x^2-2)\cos^2\left(\dfrac{1}{x-\sqrt{2}}\right) & \text{if } x \neq \sqrt{2} \\ -1 & \text{if } x = \sqrt{2} \end{cases} \quad ; \quad c = \sqrt{2}$$

(b) [*8 marks*] What is meant by saying that a function $f(x)$ is *continuous* at a point c. (You may assume that f is defined on an interval (a, b) that contains c.)

Show that if $f(x)$ and $g(x)$ are both continuous at c then so is the product $f(x) \cdot g(x)$.

(c) [*6 marks*] Which of the following functions is continuous at 0:

(i) $$f(x) = \begin{cases} |x^3|/x^3 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(ii) $$f(x) = \begin{cases} |x^3|/x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(Give reasons for your answers.)

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Question 5. Let $f : [a, b] \to \mathbb{R}$ and let $a < c < b$.

(a) [*8 marks*] Define what is meant by saying that f is *differentiable* at the point c. Show carefully that if f and g are differentiable at c then the product $f \cdot g$ is differentiable at c. (You may assume without proof that if f is differentiable at c then f is continuous at c.)

(b) [*6 marks*] Prove that if f is differentiable at c and c is a local maximum or local minimum of f, then $Df(c) = 0$. Give an example to show that the converse is false.

(c) [*6 marks*] State the Mean Value Theorem and use it to show that

$$y^4 \exp y - x^4 \exp x > 5e(y - x)$$

if $1 \le x < y$, where $e = \exp 1$

Question 6. (a) [*10 marks*] Let $f : [a, b] \to \mathbb{R}$ be continuous.

(i) Explain what is meant by a partition P of the interval $[a, b]$ and what is meant by the *upper and lower sums* $U(f; P)$ and $L(f; P)$ for P.

(ii) Explain how the *Riemann integral* $\int_a^b f(x)dx$ is defined using upper and lower sums.

(b) [*10 marks*] **From the definition of the Riemann integral** show that

$$\int_0^2 x dx = 2$$

(You may assume without proof that $1 + 2 + 3 + \ldots + m = \frac{1}{2}m(m + 1)$ for any $m \in \mathbb{N}$)

Question 7. (a) [*12 marks*] Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $F(x) = \int_a^x f(t)dt + c$ for $a \le x \le b$. Show that $F(x)$ is differentiable in the interval $[a, b]$ with derivative $DF(x) = f(x)$.

(State carefully any properties of the integral that you assume.)

(b) [*8 marks*] Let $g(x) = \int_0^{x^3} \ln\left(1 + \frac{1}{2}\cos(t^2)\right) dt$. Show that g is differentiable for all x and find its derivative.

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