THE UNIVERSITY OF SWAZILAND

Department of Mathematics

Supplementary Examination 2006

M331
REAL ANALYSIS

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Throughout this paper the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ stand for the natural numbers, the integers, the rational numbers and the real numbers respectively.

Question 1. Let A be a subset of the real numbers.

(a) [*10 marks*] What is meant by saying that A is *bounded below*?

Which of the following sets is bounded below? Give reasons for your answers.

(i) $\{q^3 : q \in \mathbb{Q} \text{ and } q < 2\}$

(ii) $\left\{\dfrac{3n^2+2}{n^2-2} : n \in \mathbb{N}\right\}$

(iii) $\{2^n : n \in \mathbb{Z}\}$

(b) [*6 marks*] What is meant by $\inf A$ for a set A that is bounded below? Find $\inf A$ for each set in (a) that is bounded below.

(c) [*4 marks*] Which of the following statements is always true? Give a proof for those that are true and a counterexample for those that are false.

(i) if A is bounded below then $\mathbb{R} \setminus A$ is not bounded below (where $\mathbb{R} \setminus A = \{x \in \mathbb{R} : x \notin A\}$);

(ii) if A is bounded below then the set $-A = \{-x : x \in A\}$ is not bounded below.

Question 2. (a) [*8 marks*] Let (a_n) be a sequence of real numbers and $l \in \mathbb{R}$. Give a precise definition of the statement that

$$\lim_{n\to\infty} a_n = l$$

Show *directly from the definition* that

$$\lim_{n\to\infty} \frac{(\sqrt{n}+1)^2}{n+1} = 1.$$

(b) [*12 marks*] Which of the following sequences (a_n) is convergent? For those that are, find the limit. State clearly any facts about limits that you use.

(i) $a_n = \dfrac{5n^4 - n + 2}{13n^2 - n^3}$

(ii) $a_n = \dfrac{n + 2n^2 - 4n^3}{2n^3 - 9n}$

(iii) $a_n = \sqrt{n^2 - \frac{2}{n^2}}$

(iv) $a_n = \sqrt{n^2+1} - n$

CONT ...

Question 3. (a) [*4 marks*] Let (a_n) be a sequence. (i) Define what is meant by the *partial sums* of the series $\sum a_n$ (ii) What is meant by saying that $\sum_{n=1}^{\infty} a_n = s$

(b) [*6 marks*] Prove carefully that if $\sum_{n=1}^{\infty} a_n = s$ and $\sum_{n=1}^{\infty} b_n = t$ then $\sum_{n=1}^{\infty}(a_n + b_n) = s + t$.

(c) [*6 marks*] Show that each of the following series converges, stating any general theorems that you use.

(i) $\sum (-1)^n \frac{1}{\sqrt{n}}$

(ii) $\sum \frac{25^n}{n^n}$

(d) [*4 marks*] Show that if $\sum a_n$ and $\sum b_n$ are both convergent and $a_n \geq 0$ and $b_n \geq 0$ then $\sum a_n b_n$ is convergent.

Question 4. (a) [*6 marks*] Find $\lim_{x \to c} f(x)$ for each of the following functions and the given value of c.

(i) $f(x) = \begin{cases} \dfrac{x^4 - 25}{x^2 - 5} & \text{if } x^2 \neq 5 \\ 12 & \text{if } x^2 = 5 \end{cases}$; $c = \sqrt{5}$

(ii) $f(x) = \begin{cases} (x^2 - 4)\sin\left(\dfrac{1}{x - 2}\right) & \text{if } x \neq 2 \\ -4 & \text{if } x = 2 \end{cases}$; $c = 2$

(b) [*7 marks*] What is meant by saying that a function $f(x)$ is *continuous* at a point c. (You may assume that f is defined on an interval (a, b) that contains c.)

Prove that if $f(x)$ and $g(x)$ are both continuous at c then so is the sum $f(x) + g(x)$.

(c) [*7 marks*] Which of the following functions is continuous at 0

(i) $f(x) = \begin{cases} x\cos(1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(ii) $f(x) = [\sin x]$ (where $[z]$ is the integer part of z - that is, the greatest integer $\leq z$).

(Give reasons for your answers.)

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Question 5. Let $f : [a, b] \to \mathbb{R}$ and let $a < c < b$.

(a) [*8 marks*] Define what is meant by saying that f is *differentiable* at the point c. Show carefully that if f and g are differentiable at c then $f + g$ is differentiable at c.

(b) [*6 marks*] Prove that if f is differentiable at c and c is a local maximum or local minimum of f, then $Df(c) = 0$. Give an example to show that the converse is false.

(c) [*6 marks*] State the Mean Value Theorem. Apply it to the function $f(x) = \mathrm{e}^x \sin x$ to show that if $x < y$ then

$$|\mathrm{e}^y \sin y - \mathrm{e}^x \sin x| \leq \sqrt{2}\,\mathrm{e}^y (y - x)$$

where $e = \exp 1$.

[You may use without proof the fact that $(\sin x + \cos x)^2 = 1 + \sin 2x$].

Question 6. (a) [*10 marks*] Let $f : [a, b] \to \mathbb{R}$ be continuous. Explain how the *Riemann integral* $\int_a^b f(x)dx$ is defined using upper and lower sums.

(b) [*10 marks*] **From the definition of the Riemann integral** show that

$$\int_0^1 x dx = \frac{1}{2}$$

(You may assume without proof that $1 + 2 + 3 + \ldots + m = \frac{1}{2}m(m + 1)$ for any $m \in \mathbb{N}$)

Question 7. (a) [*12 marks*] Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $F : [a, b] \to \mathbb{R}$ is differentiable with $DF(x) = f(x)$ for $a \leq x \leq b$. Show that

$$\int_a^x f(t)dt = F(x) - F(a)$$

for each $x \in [a, b]$. (State carefully anything you assume about the differentiability of the function $G(x) = \int_a^x f(t)dt$; and state carefully any other properties of derivatives or integrals that you assume.)

(b) [*8 marks*] Let $g(x) = \int_0^{x^3} \exp(1 + 2\sin t)\, dt$. Show that g is differentiable for all x and find its derivative. (State clearly any theorems that you use.)

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