UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2006

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	DYNAMICS II
COURSE NUMBER	:	M 355
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Prove that if the transformation equations are given by $r_\nu = r_\nu(q_1, q_2, \ldots, q_n, t)$ then

$$\frac{\partial \dot{r}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial r_\nu}{\partial q_\alpha}$$

[5 marks]

(b) Suppose that the kinetic energy T does not contain the time t explicitly and that the potential V depends on $\dot{q}_\alpha$ as well as q_α. Prove that

$$\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T,$$

[5 marks]

(c) If the Hamiltonian

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$$

is expressed as a function of the generalised coordinates q_α and the momenta p_α ONLY and DOES NOT contain the time t explicitly, prove that

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad , \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

[6 marks]

(d) If $f = f(p_\alpha, q_\alpha, t)$ and H is the Hamiltonian, prove that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]$$

[4 marks]

QUESTION 2

2. Consider the system of massless pulleys connected by a light inextensible string of length l as shown in Figure 1 Taking q_1 and q_2 to be the generalized coordinates, show that the equations

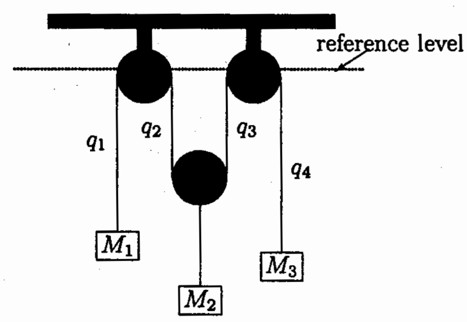

Figure 1:

of motion for the system are given by

$$\begin{aligned}(M_1 + M_3)\ddot{q}_1 + 2M_3\ddot{q}_2 &= (M_1 - M_3)g \\ 2M_3\ddot{q}_1 + (M_2 + 4M_3)\ddot{q}_2 &= (M_2 - 2M_3)g\end{aligned}$$

[20 marks]

QUESTION 3

3. (a) Find the extremal curve of

$$I = \int_0^{\frac{\pi}{2}} (y^2 - (y')^2 - 2y\sin x)dx$$

subject to the boundary conditions $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = 2$. [7 Marks]

(b) Find the extremal curve of

$$I = \int_0^{\frac{\pi}{4}} (y_1^2 + y_1'y_2' + (y_2')^2)dx$$

subject to the boundary conditions $y_1(0) = 1$ and $y_1\left(\frac{\pi}{4}\right) = 2$ and $y_2(0) = \frac{3}{2}$ and $y_2\left(\frac{\pi}{4}\right)$ is not given. [7 Marks]

(c) Use Poisson brackets to show that the following transformation is canonical [6 marks]

$$\begin{aligned}q_1 &= \sqrt{2P_1}\sin Q_1 + P_2, \quad p_1 = \frac{1}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right) \\ q_2 &= \sqrt{2P_1}\cos Q_1 + Q_2, \quad p_2 = -\frac{1}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right)\end{aligned}$$

QUESTION 4

4. (a) Given the following Lagrangian function

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}\kappa(x_1^2 + x_2^2) - \frac{1}{2}\kappa(x_2 - x_1)^2$$

for a certain mechanical system.

i. Find the corresponding Hamiltonian function. [6 marks]

ii. Using Hamilton's equations obtain the equations of motion for the system. [6 marks]

(b) For a certain system the kinetic energy T and the potential energy V are given by

$$T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_1\dot{q}_2 + \dot{q}_2{}^2),$$
$$V = \frac{3}{2}q_2^2$$

where $\dot{q}_1, \dot{q}_2$ are the generalised coordinates. Write down Lagrange's equations and hence deduce an expression for $\dot{q}_2$ in terms of t. [8 marks]

QUESTION 5

5. Use the following definition

$$[F, G] = \sum_\alpha \left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha} \right)$$

of a Poisson bracket between two physical quantities $F(q_\alpha, p_\alpha, t)$ and $G(q_\alpha, p_\alpha, t)$ to prove the following properties.

(a) $[u, v] = -[v, u]$ [3 marks]

(b) $[u, u] = 0$ [2 marks]

(c) $[u, v + w] = [u, v] + [u, w]$ [3 marks]

(d) $[u, vw] = v[u, w] + [u, v]w$ [3 marks]

(e) $[q_\alpha, p_\beta] = \delta_{\alpha\beta}$ [3 marks]

(f) $\dot{q}_\alpha = [q_\alpha, H]$ [3 marks]

(g) $\dot{p}_\alpha = [p_\alpha, H]$ [3 marks]

where q_α are generalized coordinates, p_α are generalized momenta, H is the Hamiltonian function and $\delta_{\alpha\beta}$ is the kronecker delta.

QUESTION 6

6. Use the Beltrami identity ($F - y'\frac{\partial F}{\partial y'}$ = Constant) to show that the extremum for the integral

$$I = \int_0^a \sqrt{\frac{1+y'^2}{2y}}\,dx$$

satisfies the differential equation

$$y' = \sqrt{\frac{2c-y}{y}}.$$

By making the substitution $y = 2c\sin^2\theta$, show that the solution of the differential equation is $x = c(2\theta - \sin 2\theta)$ [20 marks]

QUESTION 7

7. (a) The Hamiltonian of a two-dimensional harmonic oscillator of unit mass is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2)$$

where ω is a constant. Given that

$$A = q_1p_2 - q_2p_1 \quad \text{and} \quad B = \omega q_1 \sin\omega t + p_1 \cos\omega t$$

Show that both A and B are constants of motion. [10 marks]

(b) A particle moves in the xy plane subject to the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\Omega}{2}(-\dot{x}y + \dot{y}x)$$

where Ω is a constant.

i. Write down the Hamiltonian function for the system. [5 marks]

ii. Using the Hamiltonian, find the equations of motion for the system. [5 marks]