UNIVERSITY OF SWAZILAND

Final Examination 2006

Title of Paper : Numerical Analysis II

Program : BSc./B.Ed./B.A.S.S. IV

Course Number : M 411

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of seven (7) questions on THREE (3) pages.
2. Answer any five (5) questions.
3. Non-programmable calculators may be used.

Special Requirements : None

THIS EXAMINATION PAPER MAY NOT BE OPENED UNTIL PERMISSION TO DO SO IS GRANTED BY THE INVIGILATOR.

Question 1

(a) Let $f(x) = \cos^{-1} x$ for $-1 \le x \le 1$. Find the polynomial of degree 1, $p_1(x) = a_0 + a_1 x$, which minimizes

$$\int_{-1}^{1} \frac{[f(x) - p_1(x)]^2}{\sqrt{1-x^2}}\, dx.$$

[8 marks]

(b) (i) Show that the family of trigonometric functions $\{\phi_j(x)\}_{j=0}^{\infty}$, where $\phi_j(x) = \cos jx$, is orthogonal on $[0,\ 2\pi]$ with respect to the weight function $w(x) \equiv 1$. Also calculate $\|\phi_j(x)\|$.

[8 marks]

(ii) Hence, for $f(x) \in C[0,\ 2\pi]$, determine a_0 and a_1 such that $p_1 = a_0 + a_1 \cos x$ minimizes

$$\int_0^{2\pi} [f(x) - p_1(x)]^2\, dx.$$

[6 marks]

Question 2

(a) Compute the linear minimax approximation to $f(x) = \sqrt{1+x^2}$ on [0, 1].

[10 marks]

(b) It is known that the variables l and F are related by the equation

$$F = k(l - 5.3),$$

where k is a physical constant. If measurements of l and F are made as given in the following table, use the least squares technique to obtain an approximation for k.

l	F
7.0	2.0
9.4	4.0
12.3	6.0

[10 marks]

Question 3

(a) For the initial value problem

$$y' = -y + x + 1, \qquad y(0) = 1,$$

construct a Taylor's series method of order 3, and hence approximate the solution at $x = 0.2$ taking $h = 0.1$.

[12 marks]

(b) (i) Convert the initial value problem

$$y'' - 2y' + y = xe^x - x, \quad y(0) = y'(0) = 0,$$

to a first order system, with appropriate initial conditions.

[3 marks]

(ii) Perform one step of the Modified Euler method to the system in (i) above to obtain an approximation to the solution at $x = 0.2$ by taking $h = 0.2$.

[5 marks]

Question 4

(a) Discuss consistency, stability and convergence of the linear multistep method

$$y_{n+3} = y_{n+2} + \frac{h}{24}\left[55f_{n+2} - 59f_{n+1} + 37f_n - 9f_{n-1}\right].$$

[8 marks]

(b) Use the finite difference method to approximate the solution to the two-point boundary value problem

$$y'' = 4(y-x), \qquad 0 \le x \le 1, \quad y(0) = 0, \quad y(1) = 2.$$

Take $h = \frac{1}{3}$, and compare your results against the analytic solution of

$$y(x) = x + e^2(e^4 - 1)^{-1}(e^{2x} - e^{-2x}).$$

[12 marks]

Question 5

(a) Assuming real λ, find the interval of absolute stability of the multistep method

$$y_{n+1} = y_n + \frac{h}{2}\left[f_{n+1} + f_n\right],$$

and hence conclude whether or not it is A-stable.

[6 marks]

(b) Determine the order of the multistep method

$$y_{n+4} = y_n + \frac{4h}{3}\left[2f_{n+3} - f_{n+2} + 2f_{n+1}\right],$$

and find the leading term in the local truncation error.

[8 marks]

(c) Find a general formula for all two-step third-order methods.

[6 marks]

Question 6

Consider the following hyperbolic partial differential equation:

$$U_{tt} - U_{xx} = 0, \quad 0 \le x \le 1, \quad t > 0;$$
$$U(0,t) = 0, \quad U(1,t) = 0, \quad t > 0,$$
$$U(x,0) = 0, \quad U_t(x,0) = 0, \quad 0 \le x \le 1$$

By using the finite-difference method with $h = k = 1/3$, approximate the solution at $t = 2/3$ of the PDE, comparing your results with the actual solution of $U(x,t) = \cos \pi x \sin \pi t$ at $t = 2/3$. Assume the differential equation holds on the initial line, and use a second order Taylor series approximation to find an
approximation to the solution at $t = 1/3$.

[20 marks]

Question 7

(a) Consider the parabolic differential equation

$$U_t - U_{xx} = 0, \quad 0 \le x \le 1, \quad t > 0;$$
$$U(0,t) = 0, \quad U(1,t) = 0, \quad t > 0,$$
$$U(x,0) = \sin \pi x, \quad 0 \le x \le 1$$

(i) Construct a numerical method using the forward difference quotient based at (x_i, t_j) to approximate U_t and the usual central-difference quotient to approximate U_{xx}. Show that the resulting difference problem has the matrix representation

$$\mathbf{U}^{(n+1)} = A\mathbf{U}^{(n)},$$

defining the matrix A appropriately.

[8 marks]

(ii) Using $h = k = 1/3$, determine whether the method is stable.

[4 marks]

(b) Determine the coefficients a, b and c so that the LMM

$$y_{n+1} = ay_n + h(bf_{n+1} + cf_n)$$

is order 2.

[8 marks]

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