UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2006

B.Sc. IV, B.Ed. IV, B.A.S.S. IV

TITLE OF PAPER: Numerical Analysis II

COURSE NUMBER: M411

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

1. This paper consists of SEVEN questions on THREE pages.
2. Answer ANY FIVE questions

SPECIAL REQUIREMENTS: NONE.

THIS EXAMINATION PAPER MUST NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

1. (a) It is known that the variables x and y are related by the equation

$$y = k(x - 5.3)$$

where k is a physical constant. If points (2,7), (4,9.4), and (6,12.3) are observed, use the least squares technique to obtain an approximation for k.

[10 marks]

(b) For Chebyshev polynomials, prove each of the following statements:

$$(T_i, T_j) = \begin{cases} 0, & i \neq j \\ \pi, & i = j = 0 \\ \frac{\pi}{2}, & i = j > 0 \end{cases}$$

[10 marks]

2. (a) Show that the family of trigonometric functions $\{\phi_j(x)\}_{j=0}^{\infty}$, where $\phi_j(x) = \sin jx$, is orthogonal on $[0, 2\pi]$ with weight function $w(x) \equiv 1$. Also calculate $\|\phi_j\|$.

[10 marks]

(b) Hence, for $f(x) \in C[0, 2\pi]$, determine a_0 and a_1 such that $p_1 = a_0 + a_1 \sin x$ minimizes

$$\int_0^{2\pi} [f(x) - p_1(x)]^2 \, dx.$$

[10 marks]

3. (a) Convert the initial value problem

$$y''' + 2y'' - y' - 2y = e^x, \qquad y(0) = 1, \;\; y'(0) = 2, \;\; y''(0) = 0,$$

to a first-order system $\mathbf{u}' = \mathbf{f}(x, \mathbf{u})$.

[8 marks]

(b) Apply the modified Euler method

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\left\{f(x_n, y_n) + f(x_{n+1}, \overline{y}_{n+1})\right\}, \\ \overline{y}_{n+1} &= y_n + hf(x_n, y_n) \end{aligned}$$

to the system in (a) with $h = 0.1$ to obtain an approximation to $\mathbf{u}(0.1)$.

[12 marks]

4. (a) Use the Gram-Schmidt procedure to calculate L_1 and L_2, where $\{L_o, L_1, L_2\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ and $L_o(x) = 1$.

[10 marks]

(b) Discuss consistency, zero-stability and convergence of the linear multistep method

$$y_{n+2} = 2y_n - y_{n+1} + \frac{h}{2}\{5f_{n+1} + f_n\}.$$

[10 marks]

5. (a) Consider the standard initial value problem $y' = f(x, y)$, $y(0) = y_0$. We would like to construct a numerical method from the quadratic interpolant $P_2(x)$, of f at the equally spaced nodes x_{n+1}, x_{n+2} and x_{n+3}.

(a) Write down the Newton form of P_2 in forward difference form.

[5 marks]

(b) By integrating between x_n and x_{n+4}, derive the method

$$y_{n+4} = y_n + \frac{4h}{3}\{2f_{n+3} - f_{n+2} + 2f_{n+1}\}$$

[10 marks]

(c) Prove that this method is order 4, and find the leading term in the local truncation error.

[5 marks]

6. Consider the following Poisson equation over the square region $\{(x, y) : 0 < x < 1, \ 0 \le y \le 1\}$:

$$u_{xx} + u_{yy} = x$$

$$u(x, 0) = u(x, 1) = \frac{1}{6}x^3; \qquad 0 \le x \le 1;$$

$$u(0, y) = 0, \quad u(1, y) = \frac{1}{6}; \qquad 0 \le y \le 1$$

(a) Using $h = k = 1/3$, write down the corresponding difference problem based on the five-point formula, stating the boundary conditions in terms of the mesh points.

[5 marks]

(b) Determine the system of equations to be used to solve the problem.

[15 marks]

7. Consider the initial-boundary problem for the parabolic differential equation

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \le x \le 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$
$$u(x,0) = \cos 2\pi x, \quad 0 \le x \le 1$$

(a) If an $O(k^2 + h^2)$ numerical method is constructed using the central difference quotient to approximate u_t and the usual difference quotient to approximate u_{xx}, show that the resulting difference problem is

$$u_{i,j+1} = u_{i,j-1} + 2\lambda(u_{i+1,j} - u_{i,j} + u_{i-1,j}),$$

defining λ appropriately.

[8 marks]

(b) Give a sketch of the configuration of points involved in the computation of the solution at an internal grid point (x_i, t_j).

[4 marks]

(c) Show that the method has the matrix form

$$\mathbf{u}^{(j+1)} = \mathbf{u}^{(j-1)} + A\mathbf{u}^{(j)} \quad \text{for each} \quad j = 0, 1, 2, \ldots$$

where $\mathbf{u}^{(j)} = (u_{1,j}, u_{2,j}, \ldots, u_{m-1,j})^T$ and A is a tridiagonal matrix.

[8 marks]

******** END OF EXAMINATION ********