UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2006

BSc. / BEd. / B.A.S.S. IV

TITLE OF PAPER	:	PARTIAL DIFFERENTIAL EQUATIONS
COURSE NUMBER	:	M 415
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Reduce the following partial differential equation

$$u_{xx} + 2u_{xy} + 3u_{yy} + 4u = 0$$

into canonical form. [14 marks]

(b) Determine the region in which the partial differential equation

$$xu_{xx} + u_{yy} = x^2$$

is hyperbolic or parabolic. [6 marks]

QUESTION 2

2. Use Laplace transformations to solve the following partial differential equations subject to the given boundary and initial conditions

(a)

$$\begin{aligned} xu_x + u_t &= xt \\ u(x,0) &= 0 \\ u(0,t) &= 0 \end{aligned}$$

[8 marks]

(b)

$$\begin{aligned} u_{xx} - \frac{1}{c^2}u_{tt} + k\sin\pi x &= 0 \qquad 0 < x < 1\ , \quad t > 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= 0 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$

[12 marks]

QUESTION 3

3. Use separation of variables to solve the following time-dependent non-homogeneous heat equation.

$$\begin{aligned} u_t &= u_{xx} + e^{-t} \qquad 0 < x < \pi \\ u(x,0) &= \cos 2x \\ u_x(0,t) &= 0 \\ u_x(\pi,t) &= 0 \end{aligned}$$

[20 marks]

QUESTION 4

4. Use Laplace Transforms to prove that the solution of the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad 0 < x < 1\,, \quad t > 0 \\ u(x,0) &= \sin 5\pi x + 2\sin 7\pi x \\ u_t(x,0) &= 0 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$

is given by

$$u(x,t) = \sin 5\pi x \cos 5c\pi t + 2\sin 7\pi x \cos 7\pi ct$$

[20 marks]

QUESTION 5

5. Use separation of variables to show that the solution of the wave equation given by

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \qquad 0 < x < L \\ u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t \right\} \sin \frac{n\pi}{L} x$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx \quad \text{and} \quad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi}{L} x dx$$

[20 marks]

QUESTION 6

6. (a) Show that the solution to the wave equation

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t)\ , \qquad -\infty < x, \infty\ , \quad t > 0, \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

may be given in the D'Alembert form; [15 marks]

$$u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

(b) Use the D'Alembert formula to solve the wave equation when [5 marks]
$f(x) = x$ and $g(x) = \cos x$

QUESTION 7

7. Use separation of variables to solve the heat equation

$$u_t = u_{xx} \qquad 0 < x < 1 \ , \qquad t > 0$$

subject to

$$\begin{aligned} u(x,0) &= 6 + 4\cos 3\pi x \\ u_x(0,t) &= 0 \\ u_x(1,t) &= 0 \end{aligned}$$

[20 marks]