THE UNIVERSITY OF SWAZILAND

Department of Mathematics

Final Examination 2006

M431
METRIC SPACES

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1. (a)[*10 marks*] What is meant by saying that (X, d) is a *metric space*?

Let d be the function defined on $\mathbb{R}^2$ by

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Prove carefully that $(\mathbb{R}^2, d)$ is a metric space.

(b) [*10 marks*] Describe the *uniform metric* and the L_1-*metric* on the set $C[a, b]$ of continuous functions defined on the interval $[a, b]$.

Let $x(t) = t^4$ and $y(t) = t^3$ for $-1 \leq t \leq 1$. Calculate the distance between x and y in $C[-1, 1]$

(i) in the uniform metric;

(ii) in the L_1-metric.

Question 2. (a) [*2 marks*] Let (X, d) be a metric space and (x_n) be a sequence in X. What is meant by saying that (x_n) is *convergent*?

(b)[*4+4 marks*] Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$. Give reasons for your answers.

(i) $x_n = \left(n^{1/n}, \sin\left(\left(n + \frac{1}{n}\right)\pi\right)\right)$ (ii) $x_n = (1 + \exp(-n), (-1)^n \cos(2n\pi))$

(c) [*10 marks*] . Explain what is meant by *pointwise convergence* of a sequence (x_n) in $C[a, b]$. Prove that if (x_n) converges to x in $C[a, b]$ in the uniform metric then (x_n) converges to x pointwise.

Let x_n in $C[0, 1]$ be defined by

$$x_n(t) = \begin{cases} \dfrac{nt}{n-1} & \text{if } 0 \leq t \leq 1 - \dfrac{1}{n} \\ n(1-t) & \text{if } 1 - \dfrac{1}{n} \leq t \leq 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that (x_n) converges pointwise to the function

$$x(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t = 1. \end{cases}$$

Deduce that (x_n) is not convergent in $C[0, 1]$ in the uniform metric.

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Question 3. (a) [*5 marks*] Define what is meant by

(i) a *Cauchy sequence* in a metric space,

(ii) a *complete metric space*.

Show that the set $(0, 1)$ with the usual metric is incomplete.

(b) [*6 marks*] Let (X, d) be a metric space with the discrete metric

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y. \end{cases}$$

Show that any Cauchy sequence in X is eventually constant, and deduce that (X, d) is complete.

(c) [*9 marks*] Explain what is meant by a *contraction* of a metric space, and state without proof the *Contraction Mapping Theorem*.

Show that the mapping $f : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \frac{1}{12}(x^5 - 2x^3 + 8)$$

is a contraction, and deduce that there is a unique solution to the equation $x^5 - 2x^3 - 12x + 8 = 0$ in the interval $[-1, 1]$.

Question 4. (a) [*6 marks*] Let (X, d) be a metric space and let $A \subseteq X$. What is meant by saying that A is *closed*? Show that if $(A_i)_{i\in I}$ is any collection of closed sets then the intersection $\bigcap_{i\in I} A_i$ is also closed.

(b)[*8 marks*] What is meant by the *closed ball* $B[a, r]$ in a metric space? By drawing a diagram or otherwise describe the closed ball $B[a, 2]$ in $\mathbb{R}^2$, where $a = (1, 1)$

(i) with the usual metric;

(ii) with the max metric.

(c) [*6 marks*] Which of the following sets A is closed in the given metric space X.

(i) $X = \mathbb{R}^2$ (with the usual metric); $A = \{(a, b) : a + b \geq 0\}$

(ii) $X = C[0, 1]$ with the uniform metric; $A = \{x : x(0) < 2\}$

(iii) $X = \mathbb{R}$ with the usual metric; $A = \{1, 2, 3, 4, \ldots\}$.

Give reasons for your answers.

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Question 5. (a) [*8 marks*] Taking the definition that a set is *open* if its complement is closed, show that A is open if and only if for every $a \in A$ there is $r > 0$ such that the open ball $B(a,r) \subseteq A$.

By considering the point $a(t) \equiv 0$ in $C[0,1]$ deduce that the set $A = \{x : x(\frac{1}{2}) = 0\}$ is not open in $C[0,1]$ with the uniform metric.

(b) [*3 marks*] Let X be a set and d_1 and d_2 be metrics on X. What is meant by saying that the metrics d_1 and d_2 are *equivalent*.

(c) [*5 marks*] Suppose that there are positive constants k, K such that

$$kd_1(x,y) \leq d_2(x,y) \leq Kd_1(x,y)$$

for all $x, y \in X$. Show that d_1 and d_2 are equivalent.

(d) [*4 marks*] Show that on $\mathbb{R}^2$ the usual (Euclidean) metric and the max metric are equivalent.

Question 6. (a) [*8 marks*] Let $f : X \to Y$, where X and Y are metric spaces. Give the definition of f is *continuous* in terms of convergence of sequences.

Show that if f is continuous and U is a closed subset of Y then $f^{-1}(U)$ is a closed subset of X.

(b) [*12 marks*] Let f be the function $f : C[0,1] \to \mathbb{R}$ defined for $x \in C[0,1]$ by

$$f(x) = \min\{x(t) : 0 \leq t \leq 1\}$$

(i) Sketch the following functions $x_n(t)$ and find $f(x_n)$ for each n.

$$x_n(t) = \begin{cases} 1 - nt & \text{if } \ 0 \leq t \leq \dfrac{1}{n} \\[2ex] nt - 1 & \text{if } \ \dfrac{1}{n} \leq t \leq \dfrac{2}{n} \\[2ex] 1 & \text{if } \ \dfrac{2}{n} \leq t \leq 1 \end{cases}$$

(ii) Show that $x_n \to x$ in the L_1 metric on $C[0,1]$ where $x(t) = 1$ for all t.

(iii) Deduce that f is not continuous with respect to the L_1 metric on $C[0,1]$ (and the usual metric on $\mathbb{R}$).

Question 7. (a) [*4 marks*] Let X be a metric space and $A \subseteq X$. What is meant by saying that (i) A is *bounded* and (ii) A is *compact*?

(b) [*6 marks*] Show that a compact set is complete and bounded.

(c) [*4 marks*] Show that if X is a metric space with the discrete metric then any infinite set $A \subseteq X$ is bounded but not compact.

(d) [*6 marks*] Which of the following sets is compact ?

(i) $\{(x,y) : 0 \leq x \leq y \leq 1\}$ in $\mathbb{R}^2$

(ii) $\mathbb{Q} \cap [0,1]$ in $\mathbb{R}$.

Give reasons for your answers.

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