THE UNIVERSITY OF SWAZILAND

Department of Mathematics

Supplementary Examination 2006

M431
METRIC SPACES

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1. (a)[*10 marks*] What is meant by saying that (X, d) is a *metric space*?

Let d be the function defined on $\mathbb{R}^2$ by

$$d(x,y) = \max\{2|x_1 - y_1|, 3|x_2 - y_2|\}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Prove carefully that $(\mathbb{R}^2, d)$ is a metric space.

(b) [*10 marks*] Describe the *uniform metric* and the *L_2-metric* on the set $C[a, b]$ of continuous functions defined on the interval $[a, b]$.

Let $x(t) = t^2$ and $y(t) = |t|$ for $-1 \leq t \leq 1$. Calculate the distance between x and y in $C[-1, 1]$

(i) in the uniform metric;

(ii) in the L_2-metric.

Question 2. (a) [*2 marks*] Let (X, d) be a metric space and (x_n) be a sequence in X. What is meant by saying that (x_n) is *convergent*?

(b)[*4+4 marks*] Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$. Give reasons for your answers.

(i) $x_n = \left(\frac{n^2+1}{2n^2+1}, \sin\left(\left(2n + \frac{1}{n}\right)\pi\right)\right)$ (ii) $x_n = (2^{-n}, \cos(n\pi))$

(c) [*10 marks*] . Explain what is meant by *pointwise convergence* of a sequence (x_n) in $C[a, b]$. Prove that if (x_n) converges to x in $C[a, b]$ in the uniform metric then (x_n) converges to x pointwise.

Let x_n in $C[0, 1]$ be defined by

$$x_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq t \leq 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that (x_n) converges pointwise to the function

$$x(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } 0 < t \leq 1. \end{cases}$$

Deduce that (x_n) is not convergent in $C[0, 1]$ in the uniform metric.

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Question 3. (a) [*5 marks*] Define what is meant by a *Cauchy sequence* in a metric space. Prove that in a general metric space a Cauchy sequence is bounded, and that a convergent sequence is Cauchy.

(b) [*6 marks*] Define what is meant by a *complete metric space*.

Show that the set $\{x : d(x, 0) \leq 1\}$ in $\mathbb{R}^2$ with the usual metric is complete whereas the set $\{x : d(x, 0) < 1\}$ is incomplete. (You may assume that $\mathbb{R}^2$ is complete.)

(c) [*9 marks*] Explain what is meant by a *contraction* of a metric space, and state without proof the *Contraction Mapping Theorem.*

Show that the mapping $f : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \frac{1}{14}(x^4 - 3x^3 + 9)$$

is a contraction, and deduce that there is a unique solution to the equation $x^4 - 3x^3 - 14x + 9 = 0$ in the interval $[-1, 1]$.

Question 4. (a) [*6 marks*] Let (X, d) be a metric space and let $A \subseteq X$. What is meant by saying that A is *closed*? Show that if $A_1, A_2, \ldots, A_n$ is a finite collection of closed sets then the union $\bigcup_{i=1}^{n} A_i$ is also closed.

(b)[*8 marks*] What is meant by the *closed ball* $B[a, r]$ in a metric space? Show that a closed ball is closed.

By drawing a diagram or otherwise describe the closed ball $B[a, 3]$ in $\mathbb{R}^2$, where $a = (4, 3)$

(i) with the Chicago metric;

(ii) with the max metric.

(c) [*6 marks*] Which of the following sets A is closed in the given metric space X.

(i) $X = \mathbb{R}^2$ (with the usual metric); $A = \{(a, b) : a + b = 0\}$

(ii) $X = C[0, 1]$ with the uniform metric; $A = \{x : x(\frac{1}{2}) < 2\}$

(iii) $X = \mathbb{R}$ with the usual metric; $A = \mathbb{Z}$.

CONT ...

Question 5. (a) [*8 marks*] Taking the definition that a set is *open* if its complement is closed, show that A is open if and only if for every $a \in A$ there is $r > 0$ such that the open ball $B(a, r) \subseteq A$.

By considering the point $a(t) \equiv 1$ in $C[-1, 1]$ deduce that the set $A = \{x : x(0) = 1\}$ is not open in $C[-1, 1]$ with the uniform metric.

(b) [*3 marks*] Let X be a set and d_1 and d_2 be metrics on X. What is meant by saying that the metrics d_1 and d_2 are *equivalent.*

(c) [*5 marks*] Suppose that there are positive constants k, K such that

$$kd_1(x, y) \leq d_2(x, y) \leq Kd_1(x, y)$$

for all $x, y \in X$. Show that d_1 and d_2 are equivalent.

(d) [*4 marks*] Show that on $\mathbb{R}^2$ the usual (Euclidean) metric and the Chicago metric are equivalent.

Question 6. (a) [*8 marks*] Let $f : X \to Y$, where X and Y are metric spaces. Give the definition of f is *continuous* in terms of convergence of sequences.

Show that if f is continuous and U is an open subset of Y then $f^{-1}(U)$ is an open subset of X. (State clearly the definition of open that you are using.)

(b) [*12 marks*] Let f be the function $f : C[-1, 1] \to \mathbb{R}$ defined for $x \in C[-1, 1]$ by

$$f(x) = x(0)$$

(i) Sketch the following functions $x_n(t)$ and find $f(x_n)$ for each n.

$$x_n(t) = \begin{cases} n^2t^2 & \text{if } \ 0 \leq |t| \leq \dfrac{1}{n} \\ \\ 1 & \text{if } \ \dfrac{1}{n} \leq |t| \leq 1 \end{cases}$$

(ii) Show that $d_{L_1}(x_n, x) = \frac{4}{3}n^{-1}$, where $x(t) = 1$ for all t. Deduce that $x_n \to x$ in the L_1 metric.

(iii) Deduce that f is not continuous with respect to the L_1 metric on $C[-1, 1]$ (and the usual metric on $\mathbb{R}$).

Question 7. (a) [*4 marks*] Let X be a metric space and $A \subseteq X$. What is meant by saying that (i) A is *bounded* and (ii) A is *compact*?

(b) [*6 marks*] Show that a compact set is closed and bounded.

(c) [*4 marks*] Show that in any metric space X any finite set $A \subseteq X$ is compact.

(d) [*6 marks*] Which of the following sets is compact ?

(i) $\{(x, y) : 0 \leq x + y \leq 1\}$ in $\mathbb{R}^2$

(ii) $\{(x, y) : 0 \leq x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$

Give reasons for your answers.

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