UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2006

BSc. / BEd. / B.A.S.S. IV

TITLE OF PAPER	:	FLUID DYNAMICS
COURSE NUMBER	:	M 455
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Write the equation of continuity for a fluid of density ρ moving with velocity $\mathbf{q}$. [2 marks]

   (b) What are the two main characteristics of potential flow? [2 marks]

   (c) Which two methods are mainly used to describe fluid motion? [2 marks]

   (d) Write down the equation for total acceleration of a fluid particle. [2 marks]

   (e) Define a streamline. [4 marks]

   (f) Find the vorticity and rotation for a fluid motion with velocity $\mathbf{q} = (-ay, ax, 0)$ where a is constant. [4 marks]

   (g) A velocity field is given by $\mathbf{q} = x\mathbf{i} - y\mathbf{j}$;
   Obtain an equation for the streamlines in the xy plane. [4 marks]

QUESTION 2

2. (a) Verify that the velocity field $\mathbf{q} = x^2\mathbf{i} - 2xy\mathbf{j}$ could describe the flow of an incompressible fluid. If the fluid is inviscid and the body force per unit mass is $\mathbf{F} = -xy^2\mathbf{i} + x^2y\mathbf{j}$, show that the pressure distribution is given by [10 marks]

$$p = p_0 - \frac{\rho}{2}x^2(x^2 + y^2)$$

   (b) Fluid flows out of a circular tank of radius A though a small circular hole of radius a located in the bottom of the tank. Assuming that the flow is steady and that the pressure both at the free surface and the exit hole is atmospheric, show that the time T required to empty the tank is given by [10 marks]

$$T = \left[\frac{A^4}{a^4} - 1\right]^{\frac{1}{2}} \left(\frac{2h_0}{g}\right)^{\frac{1}{2}}$$

   where the height $h = h_0$ when $t = 0$ and g is acceleration due to gravity.

QUESTION 3

3. (a) At a point in steady, incompressible fluid having cylindrical coordinates (r, θ, z) the velocity components are

$$(r^2 \cos\theta, -3r^2 \sin\theta, 0).$$

Determine whether or not the equation of continuity is satisfied, and if so, find the equations of streamlines. [10 marks]

(b) Consider the motion of a fluid between two parallel plates. If the lower surface is held fixed and coincides with the surface $y = 0$ while the upper surface, located at $y = 1$ is moved with a constant speed α in a plane parallel to the x-axis. Assuming that the motion of the fluid is predominantly parallel to the x-axis, i.e $\mathbf{q} = (\bar{u}(y), 0, 0)$.

i. Determine the velocity profile $\bar{u}(y)$ of the fluid motion. [10 marks]

QUESTION 4

4. Consider the boundary layer equations in the form [20 marks]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - au \qquad (2)$$

with boundary conditions

$$u = ax \ , \quad v = -(\nu a)^{\frac{1}{2}} \quad \text{on y} = 0 \quad \text{and} \quad u = 0 \quad \text{on} \quad y = \infty$$

Using the similarity transformation $\eta = \left(\frac{a}{\nu}\right)^{\frac{1}{2}} y$ and the stream function formulation $\psi = -x(\nu a)^{\frac{1}{2}} f(\eta)$ where a is a constant and ν is the dynamic viscosity, Show that equation (2) and the boundary conditions can be transformed into

$$f''' + ff'' - (f')^2 - f' = 0$$

$$f = 1 \quad , \quad f' = 1 \quad \text{on } \eta = 0 \quad \text{and} \quad f' = 0 \quad \text{on } \eta = \infty$$

QUESTION 5

5. (a) Find the stream function and velocity potential corresponding to the following complex velocity potential

$$w(z) = U\left(z + \frac{a^2}{z}\right) + \frac{m}{2\pi}\ln(z - 2a)$$

[7 marks]

(b) Find the stream function corresponding to the velocity field

$$q_r = \cos\theta + 2r\sin 2\theta\ , \quad q_\theta = -\sin\theta + 2r\cos 2\theta$$

[7 marks]

(c) Show that the complex velocity potential corresponding to the velocity field

$$\mathbf{q} = 3(y^2 - x^2)\mathbf{i} + 6xy\mathbf{j} \text{ is } w(z) = z^3$$

[6 marks]

QUESTION 6

6. (a) An infinite row of line vortices, each of strength $m > 0$, are placed at the points

$$z = 0, \pm a, \pm 2a, \pm 3a, \ldots, \pm na$$

where a is a real positive number. Show that the complex potential for the flow is

$$w(z) = \frac{im}{2\pi}\log\sin\frac{\pi z}{a} \quad + \quad \text{constant}$$

[12 marks]

N.B: You may use, without proof, the result

$$\sin\frac{\pi z}{a} = \frac{\pi z}{a}\left(1 - \frac{z}{a}\right)\left(1 + \frac{z}{a}\right)\left(1 - \frac{z}{2a}\right)\left(1 + \frac{z}{2a}\right)\ldots$$

(b) The velocity potential for a steady incompressible, irrotational flow with circulation around a fixed cylinder of radius a is given in cylindrical polar coordinates by

$$\phi = Ur\left(1+\frac{a^2}{r^2}\right)\cos\theta + \frac{k\theta}{2\pi}$$

where U is the uniform speed at infinity. Find the corresponding velocity field $\mathbf{q}$ and stream function ψ. [8 marks]

QUESTION 7

7. (a) In the $z = x + iy$ plane, a line vortex of strength $m > 0$, is placed at $z = c$ and another, of strength $-m$, at $z = -c$, where c is a real positive number. Both vortices are held fixed at these locations. Write down the complex potential w for this flow and show that the stream function ψ and the velocity potential ϕ are given by

$$\psi = \frac{m}{4\pi}\log\frac{(x-c)^2+y^2}{(x+c)^2+y^2} \quad \text{and}$$

$$\phi = -\frac{m}{2\pi}\tan^{-1}\frac{2cy}{x^2+y^2-c^2}$$

hint: You may set $z - c = r_1 e^{i\theta_1}$ and $z + c = r_2 e^{i\theta_2}$, and use $\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$ [10 marks]

(b) A container is formed by rotating the curves $z = \frac{x^2}{a}$ about the vertical z-axis. The container is filled with liquid to a depth h. At time $t = 0$ a small hole of radius $\frac{a}{n}$ (n being very large) is opened at the bottom so that the liquid drains out. Let $z(t)$ be the depth of liquid remaining at time t. Show that

$$\frac{dz}{dt} = -a\sqrt{\frac{2gz}{n^4z^2 - a^2}}$$

and hence show that, approximately

$$z(t) = \left(h^{3/2} - \frac{3at\sqrt{g}}{\sqrt{2}n^2}\right)^{2/3}$$

[10 marks]