UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2006/7

BSC./B.Ed./B.A.S.S. II

TITLE OF PAPER: Linear Algebra

COURSE NUMBER: M220

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any FIVE questions.
3. Non-programmable calculators may be us

SPECIAL REQUIREMENTS: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1

(a) Determine whether the following homogeneous system has non-trivial solutions

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 0 \\ 3x_1 + x_2 - x_3 + 2x_4 &= 0 \\ 2x_2 - x_1 + 2x_3 + 2x_4 &= 0 \end{aligned}$$

[3 marks]

(b) Evaluate the determinant using cofactor expansion along the second row

$$\begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 2 & 2 \end{vmatrix}$$

[4 marks]

(c) (i) Find the inverse of the matrix A using the Gaussian elimination algorithm on $[A : I_4]$, and then use A^{-1} to solve the system $A \cdot X = B$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 5 & -3 \\ 0 & -1 & 3 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(ii) Find a finite sequence of elementary matrices $E_1, E_2, \cdots, E_k$ such that $E_k \cdot E_{k-1} \cdots E_1 \cdot A = I_4$

[13 marks]

Question 2

Let $B = \{u_1, u_2, u_3\}$ and $B^1 = \{v_1, v_2, v_3\}$ be bases in $\mathbb{R}^3$, where
$u_1 = (1,0,0)^T \quad u_2 = (1,1,0)^T \quad u_3 = (1,1,1)^T$
$v_1 = (0,2,1)^T \quad v_2 = (1,0,2)^T \quad v_3 = (1,-1,0)^T$

(a) Find the transition matrix from B^1 to B

[7 marks]

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the basis B is

$$[T]_B = \begin{bmatrix} 3 & -6 & 9 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

. Find the matrix of T w.r. to B^1

[8 marks]

(c) Let $u \in \mathbb{R}^3$ be the vector whose coordinates relative to B are

$$[u]_B = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}.$$

Find the coordinates of u relative to B^1

[5 marks]

Question 3

(a) Solve the following system of linear equations

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 1 \\ 2x_1 + 5x_2 - 8x_3 + 6x_4 &= 4 \\ x_1 + 4x_2 - 7x_3 + 2x_4 &= 8 \end{aligned}$$

[8 marks]

(b) For which k does the following system have nontrivial solutions

$$\begin{aligned} kx_1 + 2x_2 - x_3 &= 0 \\ (k+1)x_1 + kx_2 + 0x_3 &= 0 \\ -x_1 + kx_2 + kx_3 &= 0 \end{aligned}$$

[8 marks]

(c) Determine whether the given vectors are linearly independent
$u_1 = (2, 4, 0, 4, 3)^T \quad u_2 = (1, 2, -1, 3, 1)^T \quad u_3 = (-1, -2, 5, -7, 1)^T$

[4 marks]

Question 4

(a) (i) Give the definition of a basis of a vector space
(ii) Determine whether the vectors $u_1 = (1, 1, 1)$ $u_2 = (1, 2, 3)$ $u_3 = (2, -1, 1)$ form a basis for $\mathbb{R}^3$

[10 marks]

(b) Let $S = \{v_1, v_2, \cdots, v_n\}$ be a set of non-zero vectors in a vector space V. Prove that S is linearly dependent if and only if one of the vectors v_j is a linear combination of the preceding vectors in S.

[10 marks]

Question 5

(a) Use Cramer's rule to solve the following system

$$\begin{aligned} 2x + 3y - z &= 1 \\ 3x + 5y + 2z &= 8 \\ x - 2y - 3z &= -1 \end{aligned}$$

[7 marks]

(b) Compute det(A) and, if A is invertible, find $det(A^{-1})$, where

$$A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{bmatrix}$$

[6 marks]

(c) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$

Use the adjoint of A to find A^{-1}

[7 marks]

Question 6

(a) (i) State the Cayley-Hamilton Theorem

(ii) Illustrate the validity of the Cayley-Hamilton Theorem using the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

[8 marks]

(b) Prove that if a homogeneous system has more unknowns than the number of equations then it always has a non-trivial solution.

[8 marks]

(c) By inspection, find the inverses of the following elementary matrices

(i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Question 7

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y \\ 2x + 2y \\ x + y \end{pmatrix}$$

Find the matrix of T
(i) with respect to the standard basis
(ii) with respect to B^1 and B where
$B^1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and
$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

[10 marks]

(b) Let V be all ordered pairs of real numbers. Define addition and scale multiplication as follows
$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and
$\alpha(x_1, y_1) = (\alpha x_1 + \alpha - 1, \alpha y_1 + \alpha - 1)$
show that V is vector space

[10 marks]

******** *END OF EXAMINATION* ********

(ii) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Question 7

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y \\ 2x + 2y \\ x + y \end{pmatrix}$$

Find the matrix of T
(i) with respect to the standard basis
(ii) with respect to B^1 and B where
$B^1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and
$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

[10 marks]

(b) Let V be all ordered pairs of real numbers. Define addition and scale multiplication as follows
$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and
$\alpha(x_1, y_1) = (\alpha x_1 + \alpha - 1, \alpha y_1 + \alpha - 1)$
show that V is vector space

[10 marks]

******** *END OF EXAMINATION* ********