UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2007

BSC./B.Ed./B.A.S.S. II

TITLE OF PAPER: Linear Algebra

COURSE NUMBER: M220

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any FIVE questions.
3. Non-programmable calculators may be us

SPECIAL REQUIREMENTS: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1

(a) Give the definition of each of the following
(i) A vector space
(ii) An orthogonal matrix
(iii) A symmetric matrix
(iv) A skew-symmetric matrix

[10 marks]

(b) Let V be all ordered pairs of real numbers. Define addition by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and scalar multiplication by $\alpha(x_1, y_1) = (\alpha^2 x_1, \alpha^2 y)$

Determine whether V is a vector space. **[10 marks]**

Question 2

(a) Determine whether the following mappings are linear transformations

(i) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ 2x + y \end{pmatrix}$

(ii) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 1 \\ y \end{pmatrix}$

[10 marks]

(b) Prove that the set $B = \{x^2 + 1, x - 1, 2x + 2\}$ is a basis for the vector space $P_2(x)$, where $P_2(x)-$ denotes all polynomials of degree ≤ 2 and the zero polynomial.

[10 marks]

Question 3

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x+2y+3z \end{pmatrix}$

(i) Find the standard matrix for the transformation T.
(ii) Find the matrix relative to the $R-$ bases

$$B = \{u_1 = (1,1,0), u_2 = (0,1,1), u_3 = (0,0,1)\}$$

and

$$B^1 = \{v_1 = (1,2), v_2 = (1,3)\}$$

for $\mathbb{R}^2$ respectively

[5 marks]

(b) Verify the Cayley-Hamilton Theorem for the following matrix
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{pmatrix}$$

Question 4

(a) Find the eigen values and eigenvectors of
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

[10 marks]

(b) Let $S = \{u_1, u_2, u_3, u_4\}$ where
$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix} \quad u_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
Show that the set S is a basis for $\mathbb{R}^4$

Question 5

(a) Solve the following systems

(i)

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 &= 3 \\ 4x_1 + 7x_2 + 7x_3 &= 1 \\ 4x_2 - 2x_1 + 5x_3 &= -7 \end{aligned}$$

(ii)

$$\begin{aligned} x_1 + 3x_2 - 2x_3 - 4x_4 &= 3 \\ 2x_1 + 6x_2 - 7x_3 - 10x_4 &= -2 \\ -x_1 - x_2 + 5x_3 + 9x_4 &= 14 \end{aligned}$$

[10 marks]

(b) (i) Show that the vector $v = \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix}$ is a linear combination of $u_1 = (4, 2, -3)^T$ $u_2 = (2, 1, -2)^T$ and $u_3 = (-2, -1, 0)^T$

[5 marks]

(ii) Show that $u_1 = (1, 1, 1, 1)^T, u_2 = (1, 0, 0, 2)^T$ and $u_3 = (0, 1, 0, 2)^T$ are linearly independent

[5 marks]

Question 6

(a) Determine whether the following has a non-trivial solution

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 2x_4 &= 0 \\ 3x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \end{aligned}$$

[4 marks]

(b) Prove that if A and B are both nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

[4 marks]

(c) Find the inverse A^{-1} of the following matrix A in two ways.
(i) using the augmented matrix $[A|I]$ **[6 marks]**
(ii) By computing a product $E_n E_{n-1} \cdots E_2 E_1$ elementary matrices **[6 marks]**

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{pmatrix}$$

Question 7

(a) Let $B^1 = \{v_1, v_2, v_3\}$ and $B = \{u_1, u_2, u_3\}$ be ordered bases in $\mathbb{R}^3$ where

$$v_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Find the transition matrix from B^1 to B

[10 marks]

(b) Prove that if a homogeneous system has more unknowns than the number of equations, then it always has a non-trivial solution.

[10 marks]

******** *END OF EXAMINATION* ********