UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Write the negation of the following statement: "The function f of one variable $x \in \mathbb{R}$ is a *convex function* if and only if for all real numbers x and y and for all real numbers t with $0 \leq t \leq 1$, it follows that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$." [6]

(b) Which of the following statements are true?

(i) Some animals are four legged, whereas all the rest are two legged.

(ii) The square root of any integer is a non-negative real number. [6]

(c) For each of the following, write the *converse* and the *contrapositive*:

(i) If n is an integer for which n^2 is even, then n is even.

(ii) Suppose that t is an angle between 0 and π. If t satisfies $\sin(t) = \cos(t)$, then $t = \frac{\pi}{4}$. [8]

QUESTION 2

(a) Prove that the following statements are false:

(i) For all $n \in \mathbb{N}$, $n^2 - n + 87$ is a prime number.

(ii) For all $n \in \mathbb{N}$, $2n^2$ is an odd integer.

(iii) For some $n \in \mathbb{N}$, with $n \geq 2$, $n^2 + 2n$ is a prime integer. [6]

(b) Prove that if there are at least 6 people at a party, then either 3 of them knew each other before the party, or 3 of them were complete strangers before the party. [12]

(c) Show that the polynomial $p(x) = x^4 - 2x^2 - 3$ has a root that lies between $x = 1$ and $x = 2$. [4]

QUESTION 3

(a) Define an order on the set $\mathbb{Z}$. [2]

(b) Prove that there is no integer between 0 and 1. [5]

(c) Prove that a set S of positive integers which includes 1, and which includes $n+1$ whenever it includes n, must include every positive integer. [5]

(d) Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

Deduce formulae for

(i) $1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 7 + \ldots + n(2n-1)$ and

(ii) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$.

[8]

QUESTION 4

(a) Prove that the set of all primes is infinite. [10]

(b) Prove that $\sqrt{6}$ is irrational. [10]

QUESTION 5

(a) (i) Define a partial order and a total order of a set.

(ii) What is a well-ordered set?

(iii) Which of the following are well-ordered sets?

- all odd positive integers.
- all even negative integers.
- all integers greater than -7.

- all odd integers greater that 749. [10]

(b) Prove that every real number has a decimal representation. [10]

QUESTION 6

(a) (i) Define an equivalence relation. [2]

(ii) Show that the relation

$$\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 2\}$$

is an equivalence relation. What are the equivalence classes of $\mathcal{R}$? [12]

(b) (i) Define the composition $f \circ g$ of any two functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$. [2]

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = \sin x$ and $g(x) = x^2 + 2$ for all $x \in \mathbb{R}$. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$. [4]

QUESTION 7

(a) State and prove the Fundamental Theorem of Arithmetic. [12]

(b) Prove that there are infinitely many primes of the form $3k + 2$, where k is an integer. [8]

END OF EXAMINATION