UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

<u>QUESTION 1</u>

(a) Negate the following statement: "The function f of one variable is a *convex function* if and only if for all real numbers x and y and for all real numbers t with $0 \leq t \leq 1$, it follows that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$." [6]

(b) Which of the following statements are true?

(i) The square root of any integer is a non-negative real number.

(ii) If $x < 1$, then $x^2 < 1$. [6]

(c) For each of the following statements, write the converse and the contrapositive:

(i) A quadrilateral is a parallelogram if its opposite sides are parallel.

(ii) Congruent triangles are equal in area. [8]

<u>QUESTION 2</u>

(a) Prove that in any set of n distinct integers, there must be two whose difference is divisible by $n-1$. [10]

(c) Define

(i) a partially ordered set,

(ii) a totally ordered set. [7]

(d) Give an example of a partially ordered set that is not totally ordered. [3]

QUESTION 3

(a) Define an order on the set $\mathbb{Z}$. [2]

(b) Prove that there is no integer between 0 and 1. [5]

(c) Prove that a set S of positive integers which includes 1, and which includes $n+1$ whenever it includes n, must include every positive integer. [5]

(d) Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

Deduce formulae for

(i) $1.1 + 2.3 + 3.5 + 4.7 + \ldots + n(2n-1)$ and

(ii) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$.

[8]

QUESTION 4

(a) The *Fibonacci sequence* is a sequence of integers $u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$, such that $u_1 = 1$, $u_2 = 1$ and

$$u_{n+1} = u_n + u_{n-1}$$

for all $n \geq 1$. The beginning of this sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. Prove by strong induction that for all positive integers n,

$$u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

[12]

(b) Suppose that Canada Post prints only 3 cent and 5 cent stamps. Prove that it is possible to make up any postage of n cents using only 3 cent and 5 cent stamps for $n \geq 8$. [8]

QUESTION 5

(a) Let $x = 0.a_1a_2a_3\ldots$, where for $n = 1, 2, 3, \ldots$, the value of a_n is the number 0, or 1, or 2, or 3 which is the remainder on dividing n by 4. Is x rational? If so, express x as a fraction $\frac{m}{n}$ where m and n are integers with $n \neq 0$. [8]

(b) Prove that between any two different irrational numbers there is a rational number and an irrational number. [12]

QUESTION 6

(a) (i) Define an equivalence relation. [2]

(ii) Show that the relation

$$\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod 3\}$$

is an equivalence relation. What are the equivalence classes of $\mathcal{R}$? [12]

(b) (i) Define the composition $f \circ g$ of any two functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$. [2]

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = \sin x$ and $g(x) = x^2 + 2$ for all $x \in \mathbb{R}$. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$. [4]

QUESTION 7

(a) State and prove the Fundamental Theorem of Arithmetic. [12]

(b) Prove that there are infinitely many primes of the form $3k + 2$, where k is an integer. [8]

END OF EXAMINATION