UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2006/2007

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Find the outward unit normal vector to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a, b > 0,$$

at the point $P(\frac{-a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$. [10]

(b) Find the area of the surface cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the plane $z = 0$. [10]

QUESTION 2

(a) The path of a highway and an exit ramp are superimposed on a rectangular coordinate system such that the highway coincides with the x-axis. The exit ramp begins at the origin O. After following the graph of $y = -x^4/4$ from O to the point $P(1, -1/4)$, the path follows the arc of a circle in such a way that the ramp is *continuous*, *smooth*, and has *continuous curvature*. Find the equation of this circle. [10]

(b) Find a parametrization of the first-octant portion of the cone $z = \dfrac{\sqrt{x^2 + y^2}}{2}$ between the planes $z = 0$ and $z = 3$. [10]

QUESTION 3

(a) Express the following in cylindrical coordinates:

(i) gradϕ;

(ii) div$F$;

(iii) the volume element dV, and

(iv) the Jacobian. [10]

(b) Let D be the region in the xyz-space defined by the inequalities

$$1 \leq x \leq 2, \qquad 0 \leq xy \leq 2, \qquad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2 y + 3xyz) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

by applying the transformation

$$u = x, \qquad v = xy, \qquad w = 3z$$

and integrating over the appropriate region G in the uvw-space. [10]

QUESTION 4

(a) Let $\mathbf{F} = (6xy + z^3)\hat{\mathbf{i}} + (3x^2 - z)\hat{\mathbf{j}} + (3xz^2 - y)\hat{\mathbf{k}}$ be a vector field.

(i) Show that $\mathbf{F}$ is irrotational. [3]

(ii) Find $\operatorname{div}\operatorname{curl}\mathbf{F}$. [2]

(b) Let $\mathbf{u}(x, y, z) = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$ and $\mathbf{v}(x, y, z) = \dfrac{\mathbf{u}}{(x^2 + y^2)^{\frac{1}{2}}}$ be vectors in space.

(i) Compute the divergence and the curl of $\mathbf{u}$ and $\mathbf{v}$. [8]

(ii) Find the flow lines of $\mathbf{u}$ and $\mathbf{v}$. [7]

QUESTION 5

(a) Determine the directional derivative of $\phi(x, y) = \ln\sqrt{x^2 + y^2}$ at the point $(1, 0)$ in the direction of $\dfrac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}}{2\sqrt{2}}$. [6]

(b) Find a unit normal to the surface $2x^2 + 4yz - 5z^2 = -100$ at $P(2, -2, 3)$. [6]

(c) Verify that the parametric equations

$$x = \rho^2 \cos\theta, \quad y = \rho^2 \sin\theta, \quad z = \rho$$

could be used to represent the surface $x^2 + y^2 - z^4 = 0$. Hence compute the unit normal to this surface at any point. [8]

QUESTION 6

(a) If $\mathbf{A} = (3x^2 - 6yz)\hat{\mathbf{i}} + (2y + 3xz)\hat{\mathbf{j}} + (1 - 4xyz^2)\hat{\mathbf{k}}$, evaluate $\int_C \mathbf{A} \cdot \mathrm{d}\mathbf{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C:

(i) $x = t$, $y = t^2$, $z = t^3$;

(ii) the straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(1,1,1)$;

(iii) the straight line joining $(0,0,0)$ and $(1,1,1)$. [12]

(b) Verify Green's theorem in the plane for

$$\oint_C [2x\mathrm{d}x - 3y\mathrm{d}y],$$

where C is the closed curve (described in the positive direction) of the region bounded by the curves $y = x^2$ and $y^2 = x$. [8]

QUESTION 7

(a) Evaluate $\iint_S [xz^2\,\mathrm{d}y\,\mathrm{d}z + (x^2y - z^3)\,\mathrm{d}z\,\mathrm{d}x + (2xy + y^2z)\,\mathrm{d}x\,\mathrm{d}y]$, where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$

(i) by the divergence theorem (Green's theorem in space),

(ii) directly. [12]

(b) Verify Stokes' theorem for $\mathbf{A} = 3y\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. [8]

END OF EXAMINATION