UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2006/2007

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Part of a railway line (superimposed on a rectangular coordinate system) follows the line $y = -x$ for $x \leq 0$, then turns to reach the point (3,0) following a cubic curve. Find the equation of this curve if the track is *continuous*, *smooth*, and has *continuous curvature.* [10]

(b) In each of the following, find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_{\tau=0}^{t} |\mathbf{v}(\tau)| d\tau.$$

Then find the length of the indicated portion of the curve.

(i) $\mathbf{r}(t) = (4\cos t)\hat{\mathbf{i}} + (4\sin t)\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}, \qquad 0 \leq t \leq \pi/2,$

(ii) $\mathbf{r}(t) = (e^t \cos t)\hat{\mathbf{i}} + (e^t \sin t)\hat{\mathbf{j}} + e^t\hat{\mathbf{k}}, \qquad -\ln 4 \leq t \leq 0.$ [10]

QUESTION 2

(a) Find the volume of the ellipsoid

$$\begin{aligned} x &= a\rho \sin\phi \cos\theta \\ y &= b\rho \sin\phi \sin\theta \\ z &= c\rho \cos\phi, \end{aligned}$$

where $a, b, c \neq 0; \quad \rho \geq 0, \quad 0 \leq \theta \leq 2\pi \quad \text{and} \quad 0 \leq \phi \leq \pi.$ [12]

(b) Find the angle between the planes

$$x + y = 1 \qquad \text{and} \qquad 2x + y - 2z = 2.$$

[8]

QUESTION 3

(a) Find the tangent plane and the normal line to the surface $x^2y + xyz - z^2 = 1$ at the point $P_0(1, 1, 3)$. [10]

(b) Give a formula $\mathbf{F} = M(x, y)\hat{\mathbf{i}} + N(x, y)\hat{\mathbf{j}}$ for the vector field in the plane with the properties that $\mathbf{F} = \mathbf{0}$ at the origin and that at any other point (a, b) in the plane, $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction, with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$. [10]

QUESTION 4

(a) Express the following in cylindrical coordinates:

(i) gradϕ;

(ii) div$F$;

(iii) the volume element dV, and

(iv) the Jacobian. [10]

(b) Repeat (a) for spherical coordinates. [10]

QUESTION 5

(a) Find the work done in moving a particle once around an ellipse C in the xy-plane, if the ellipse has center at the origin, with semi-major axes and semi-minor axes 4 and 3, respectively. (You may assume that the ellipse is traversed in the counterclockwise direction). [10]

(b) (i) Prove that $\int_{(1,2)}^{(3,4)} [(6xy^2 - y^3)\,\mathrm{d}x + (6x^2y - 3xy^2)\,\mathrm{d}y]$ is independent of the path joining $(1,2)$ and $(3,4)$.

(ii) Evaluate the integral in (i). [10]

QUESTION 6

(a) Evaluate $\iint_S \mathbf{A}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, where $\mathbf{A} = xy\hat{\mathbf{i}} - x^2\hat{\mathbf{j}} + (x+z)\hat{\mathbf{k}}$, S is that portion of the plane $2x + 2y + z = 6$ included in the first octant, and $\hat{\mathbf{n}}$ is the unit normal to S. [10]

(b) Verify the divergence theorem for $\mathbf{A} = (2x - z)\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} - xz^2\hat{\mathbf{k}}$ taken over the region bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. [10]

QUESTION 7

(a) (i) Prove that a necessary and sufficient condition that $A_1\,\mathrm{d}x + A_2\,\mathrm{d}y + A_3\,\mathrm{d}z = \mathrm{d}\phi$ is an exact differential is that $\nabla\times\mathbf{A} = \mathbf{0}$, where $\mathbf{A} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{i}} + A_3\hat{\mathbf{i}}$.

(ii) Show that for the case in (i),

$$\int_{x_1,y_1,z_1}^{x_2,y_2,z_2} [A_1\,\mathrm{d}x + A_2\,\mathrm{d}y + A_3\,\mathrm{d}z] = \int_{x_1,y_1,z_1}^{x_2,y_2,z_2} \mathrm{d}\phi = \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1)$$

[12]

(b) Verify Stokes' theorem for $\mathbf{A} = 3y\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. [8]

END OF EXAMINATION