University of Swaziland

Final Examination, May 2007

BSc III, Bass III, BEd III

Title of Paper : Complex Analysis
Course Number : M313
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Find all fourth roots of -81 and express in the form $a + ib$. [10 marks]

(b) Consider the real function $u = x^2 + 2xy - y^2$.

(i) Show that u is harmonic. [2 marks]

(ii) Find the harmonic conjugate of u. [4 marks]

(iii) Hence find the analytic complex function $f(z) = u + iv$ and express in terms of[1] z. [4 marks]

Question 2

(a) Use complex-number methods to express $\cos^6 \theta$ in terms of cosines of multiples of θ. [8 marks]

(b) Use the theory of residues to evaluate

$$\int_{-\pi}^{\pi} \frac{\cos\theta \mathrm{d}\theta}{5 + 4\cos\theta}.$$

[12 marks]

Question 3

(a) Consider the complex function $f(z) = \dfrac{z}{z+i}$.

(i) Find the first five non-zero terms of the Taylor expansion of $f(z)$ about $z = i$. [10 marks]

(ii) Determine the radius of convergence of the series obtained in (i). [2 marks]

(b) Evaluate

$$\oint_C \bar{z}^2 \, \mathrm{d}z,$$

where C is the circle $|z - 1| = 1$, traversed positively.

---

[1]Note: Throughout this paper the variable $z = x + iy$ is complex, with real x and y, and $i^2 = -1$.

Question 4

(a) Consider the complex number $\lambda = 2ie^{\pi i/3} + 2e^{-2\pi i/3}$.

(i) Express λ in the form $a + ib$. [5 marks]

(ii) Hence state the quadrant in which λ is located and show the $|\lambda| = 2\sqrt{2}$. [5 marks]

(b) Solve for the principal value of

$$\cos z = 2,$$

and express in the form $a + ib$. [7 marks]

(c) Evaluate $\int_0^{i\sqrt{\pi}} z \sin(z^2) \mathrm{d}z$ along any path. [3 marks]

Question 5

(a) Find the Laurent expansion of $f(z) = \dfrac{-2z+3}{z^2 - 3z + 2}$ valid in the region

(i) $|z| < 1$, [4 marks]

(ii) $|z| > 2$. [4 marks]

(b) Consider the complex function $f(z) = \dfrac{25(z^2 - 2z)}{(z+1)^2(z^2+4)}$.

(i) Locate and classify all the singular points of $f(z)$. [2 marks]

(ii) Find the value of the residue of $f(z)$ at each of the singular points. [7 marks]

(iii) Hence evaluate

$$\oint_C \frac{25(z^2 - 2z)}{(z+1)^2(z^2+4)} \mathrm{d}z$$

where C is the ellipse $x^2 + 4y^2 = 4$ traversed positively.

[3 marks]

Question 6

(a) Prove that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y. \qquad [7 \text{ marks}]$$

(b) Evaluate $\int_{i}^{2-i} (3xy + iy)\mathrm{d}z$ along the straight line joining $z = i$ and $z = 2 - i$.

[7 marks]

(c) Find all values of $\omega = \ln\left(\frac{2i}{1 - i\sqrt{3}}\right)$ and express in the form $a + ib$. [6 marks]

Question 7

(a) Consider the complex function $f(z) = ze^{-\bar{z}}$.

(i) Determine the functions $u(x, y) = \text{Re}(f)$ and $v(x, y) = \text{Im}(f)$. [6 marks]

(ii) Test whether the Cauchy-Riemann equations are satisfied and hence discuss the analyticity of $f(z)$.

[6 marks]

(b) Use the theory of residues to evaluate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4 + 4}. \qquad [8 \text{ marks}]$$

.... END OF PAPER ....