University of Swaziland

Supplementary Examination, July 2007

BSc III, Bass III, BEd III

Title of Paper : Complex Analysis
Course Number : M313
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Evaluate $\dfrac{(1-i\sqrt{3})^5}{(-1+i)^{10}}$ and express in the form $a+ib$. [7 marks]

(b) Evaluate $\displaystyle\oint_\Gamma \frac{\sin 3z\,\mathrm{d}z}{z-\frac{1}{2}\pi}$ where Γ is the circle[1]

(i) $|z|=1$, [3 marks]

(ii) $|z|=2$. [3 marks]

(c) Evaluate $\displaystyle\int_{-i}^{1}(2y-i)\mathrm{d}z$ along the straight line joining $z=-i$ and $z=1$. [7 marks]

Question 2

(a) Solve $z^2+(3+2i)z+5+i=0$. [12 marks]

(b) Find the Laurent series of $f(z)=\dfrac{1}{z+z^2}$ valid in the region

(i) $0<|z|<1$, [2 marks]

(ii) $|z+1|>1$. [6 marks]

Question 3

(a) Consider the complex function $f(z)=\dfrac{\cos 2z}{1+z^2}$.

(i) Find the first four non-zero terms of the Mclaurin expansion of $f(z)$. [8 marks]

(ii) Determine the radius of convergence of the series obtained in (i). [2 marks]

[1] Note: Throughout this paper the variable $z=x+iy$ is complex, with real x and y, and $i^2=-1$.

(b) Use the method of residues to evaluate

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}.$$ [10 marks]

Question 4

(a) Consider the function $f(z) = \sin^{-1} z$.

(i) Show that $f(z) = -i \ln\left(iz + \sqrt{1-z^2}\right)$, stating any restrictions. [7 marks]

(ii) Hence derive the formula

$$\frac{d}{dz}\left(\sin^{-1} z\right) = \frac{1}{\sqrt{1-z^2}}.$$ [6 marks]

(b) Solve for the principal value of

$$\cos z = -2,$$

and express in the form $a + ib$. [7 marks]

Question 5

(a) Consider the real function $u(x, y) = x^2 - y^2$.

(i) Show that u is a harmonic function. [3 marks]

(ii) Find the harmonic conjugate of u. [6 marks]

(iii) Hence find the analytic function $f(z) = u + iv$, and express in terms of z. [3 marks]

(b) Evaluate

$$\int_{-1+i}^{1+i} \bar{z}\,dz$$

along the lower half of the unit circle. [8 marks]

Question 6

(a) Consider the complex function $f(z) = \dfrac{\sin z}{z^4 + 9z^2}$.

(i) Find all singularities of $f(z)$ and classify them. [5 marks]

(ii) Find the value of the residue of $f(z)$ at each of the singular points. [8 marks]

(iii) Evaluate the integral

$$\oint_\Gamma \frac{\sin z}{z^4 + 9z^2} \mathrm{d}z, \qquad [2 \text{ marks}]$$

where Γ is the circle $|z| = 1$ traversed positively.

(b) Find the principal value of $\omega = (-i)^i$. [5 marks]

Question 7

(a) Evaluate the integral

$$\int_0^\infty \frac{\mathrm{d}x}{4 + x^4}. \qquad [10 \text{ marks}]$$

(b) Consider the complex function $f(z) = \sin \bar{z}$.

(i) Determine the real functions $u(x, y) = \mathrm{Re}(f)$ and $v(x, y) = \mathrm{Im}(f)$. [5 marks]

(ii) Test whether the Cauchy-Riemann equations are satisfied and hence discuss the analyticity of $f(z)$. [5 marks]

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